Social Welfare Function in a Generalized Form and its Disaggregation by Components of Income

Pundarik Mukhopadhaya*

JEL Classification: D39, D69

Key Words: Social Welfare Function, Paretian, Gini, Concentration Coefficient, Relative equity

Abstract: The Social Welfare Function (SWF) is a decision rule to rank alternative social states in a complete fashion in terms of social welfare. This paper questions the philosophy of Paretian Principle as a desirable property of the SWF. It shows that it is possible to generalize the widely used Sen SWF, which can be non-Paretian under special circumstances. Also, it demonstrates the disaggregation method of this SWF by components of income using the Gini decomposition process. The method is important in benefit/cost analysis and in the determination of the target component. Thus the policy maker will be better able to assess whether an increase in social insurance, for example, in terms of health protection to all in equal proportion, or a proportionate increase in wage rate will be more effective from a social welfare point of view in the labour market.

* Department of Economics, Faculty of Arts and Social Sciences, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260; Phone: (65) 68746129, Fax: (65) 6775 2646; Email: pundarik@nus.edu.sg

Social Welfare Function in a Generalized Form and its Disaggregation by Components of Income

I Introduction

An important aim of welfare economics is the ordering of alternative social states in a complete fashion in terms of social welfare. For example, suppose there are three alternative economic situations denoted by A, B and C. We can rank these three situations according to some decision rule. One of these decision rules is called the Social Welfare Function (SWF). Economists[1] have agreed that SWF is a function of equity and efficiency. Because of its innocuous nature, the definition of efficiency in welfare economics has been widely accepted through the Pareto principle. However, this principle has some disadvantages. According to this criterion, if the rich becomes richer due to some labour market policy change, such a change is acceptable. The widely used non-utilitarian Sen SWF (Sen 1974, 1976) and its variants, being linear functions of income, can be shown to be Paretian. This paper questions the philosophy of Paretianity as a desirable property of the SWF and proposes a non-Paretian SWF in the next section.

The total income received by a household can be divided into a number of components depending on its source. Whichever way total income is disaggregated, one must be able to determine the exact contribution of each component to total welfare. As the Gini coefficient is one of the arguments of the proposed SWF, a method of Gini decomposition is used to disaggregate the SWF into components of income. The relative welfare share of each component can be obtained using this decomposition procedure. However, at times when there are high growth rates or inflation in the economy, some components (say, wage or salary) may grow comparatively faster than other components. In this case the elasticity of welfare with respect to the mean income of the component might provide a more reliable picture of relative effect on total welfare of the component. These elasticities are helpful in facilitating policy discussions about the level of welfare of the society. This is important because a large part of a household's income is derived from government benefits. Furthermore, the government can influence other sources of income using

[1] Among others see Kondor (1975).

appropriate fiscal policies. Section III demonstrates the decomposition procedure of the SWF and the method of obtaining the elasticity. Some concluding remarks are made in the last section.

II Paretian Principle and the SWF

The individualistic abbreviated SWF can be represented as a function of equity and efficiency:

(1) $W = W(S,\theta)$

where S and θ are both functions of $\boldsymbol{x}$, the income profile of the society. S is a representation of total income of the society, which captures the efficiency aspect. $\theta = \theta(x_1, x_2, \ldots, x_n)$ represents the inequality of income in the society, which, in turn, represents the equity aspect of the state. Clearly an increase (decrease) in S will increase (decrease) the social welfare of the society and increase (decrease) in θ will decrease (increase) the welfare of the society.

Thus for simplicity, and to quantify the SWF, we adopt a very simple definition of equity. The definition of efficiency in welfare economics is generally associated with the Pareto principle. Because of its relatively innocuous nature this principle is widely accepted and it is one of the fundamental properties of a SWF. The Pareto principle can be written as:

(2) $$\frac{\partial W}{\partial x_i} > 0, i = 1, \ldots, N$$

According to the Pareto principle, if there is an increase in income of one person in the society, other things remaining equal, social welfare will increase. An increase in one person's income affects social welfare in two ways: first, by increasing the total income of the society and second, by changing the inequality of the society (it could increase or decrease depending on whose income has increased). Thus, if the increase in income increases the inequality and if the effect of this increase in inequality is less than that of the increase in income ie:

(3) $$\left|\frac{\partial W}{\partial \theta}\frac{\partial \theta}{\partial x_i}\right| < \frac{\partial W}{\partial S}\frac{\partial S}{\partial x_i}$$

then the SWF will satisfy the Pareto Principle. (This is the case when the extra income goes to the person above the mean/median income. If this extra income goes to a person below the mean/median income, inequality will decrease and thus society's welfare will benefit in two ways). According to this criterion, if the rich become richer due to some policy change the change is acceptable. As an example let us consider a society with three persons having incomes \$10, \$1 and \$0. Now, suppose the government introduces a growth-oriented policy and the income profile becomes:

$$(4) \qquad \begin{pmatrix} 10 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 100 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 10000 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1000000 \\ 1 \\ 0 \end{pmatrix} \rightarrow \Lambda$$

According to any Paretian SWF, this growth process will increase the welfare of the society. Sen (1974) using a non-utilitarian approach[2] introduced axiomatically the following SWF:

$$(5) \qquad W = \mu(1\text{-}G),$$

where μ is the mean income of the society and G is the Gini coefficient of the income distribution.[3] It can be shown that the Sen SWF also follows the Paretian principle; thus according to this SWF the welfare of the above society is increasing. For the Sen SWF, the rate of substitution between inequality and efficiency at a constant welfare level can be given by:

$$(6) \qquad \frac{dG}{d\mu} = \frac{1-G}{\mu}$$

Clearly this SWF is highly sensitive to mean income and less sensitive to inequality. As both G and μ are determined by the income profile of the society and cannot be varied by the decision-maker at different levels of growth or income distribution, this SWF is quite rigid. The marginal welfare change with respect to mean income, in this case, is (1-G) which is a constant. Thus, in the case of an international comparison, this SWF will always be biased in favour of developed countries, which have

[2] All the utilitarian SWFs are Paretian.

[3] Sen (1976) shows that this index, calculated from the income distribution, 'is a sub-relation of social preference relation defined in the distribution of commodities'. Alternatively, Yitzhaki (1979, 1982) showed that this index could be based on relative deprivation. Sheshinski (1972) also derived this index from the Gini coefficient. Dagum (1990, 1993) arrived at the same SWF from a somewhat different perspective.

relatively high per capita incomes and relatively low inequality. Also, for any country the welfare status over time may not be comparable using this SWF, if that country is experiencing high growth rates. Bearing these criticisms in mind let us propose the following theorem:

Theorem 1

A more generalized class of SWF can be proposed as: $W = \mu^{\beta}(1 - G)$, which is non-Paretian if β is less than 1.

Proof:

From equation (2) we know the condition of Paretianity as $\frac{\partial W}{\partial x_i} > 0$.

Therefore we can write:

$$\frac{\partial W}{\partial x_i} = \frac{\partial}{\partial x_i}[\mu^{\beta}(1-G)]$$

$$= \frac{\partial}{\partial x_i}[(\sum \frac{x_i}{n})^{\beta}(1 - \frac{\sum(2i-n-1)x_i}{n\sum x_i})]$$

$$= \beta\mu^{\beta-1}\frac{1}{n}[1-G] + \mu^{\beta}[\frac{0-(2i-n-1)n^2\mu + n\sum(2i-n-1)x_i}{(n\sum x_i)^2}$$

$$= \frac{\frac{1}{n}\beta\mu^{\beta-1}n^4\mu^2(1-G) - n^2\mu^{\beta+1}(2i-n-1) + \mu^{\beta}n\sum(2i-n-1)x_i}{(n^2\mu)^2}$$

To satisfy Paretianity this expression has to be greater than zero – that means

$$\frac{1}{n}\beta\mu^{\beta-1}n^4\mu^2(1-G) + \mu^{\beta}n\sum(2i-n-1)x_i > n^2\mu^{\beta+1}(2i-n-1)$$

$$\Rightarrow n^3\beta\mu^{\beta+1}(1-G) + \mu^{\beta}n\sum(2i-n-1)x_i > n^2\mu^{\beta+1}(2i-n-1)$$

(7) $$\Rightarrow n\beta(1-G) + \frac{1}{n\mu}\sum(2i-n-1)x_i > 2i-n-1$$

$$\Rightarrow \beta(1-G) + \frac{\sum(2i-n-1)x_i}{n^2\mu} > \frac{2i-n-1}{n}$$

$$\Rightarrow \beta - \beta G + G > \frac{2i-n-1}{n}$$

for i=1,..,n

which is always true from the lowest income to the median income as the left-hand side of the last line of expression (7) is always positive. The SWF, $W = \mu^{\beta}(1 - G)$, is Paretian if

(8) $$\beta + G - \beta G > \frac{n-1}{n},$$ [putting the maximum value for i in (7)]

For a large n, (8) can be written as:

(9) $$\beta + G - \beta G \geq 1$$

which will never be satisfied for a value of β less than 1.

Thus we find that

(10) $$W = \mu^{\beta}(1-G), 0 < \beta < 1$$

could be a generalized class of SWF which is non-Paretian. Thus with this SWF social welfare will decrease if the benefits of the growth process fall only into the hands of the richest person in the society. When the value of β is 0 the SWF will become a function of inequality (G) only regardless of the level of efficiency of the society.

This SWF with variable values of β has certain advantages over the Sen SWF. For this SWF

(11) $$\frac{dG}{d\mu} = (\frac{1-G}{\mu})\beta$$

Here the decision-maker has the choice of β, and thus the SWF is now flexible with respect to the trade-off between efficiency and equality. If she wants to attach more importance to efficiency than equality she will choose a higher value of β (approaching one), and on the contrary if she is an equity-lover she will set a lower value of β.

This SWF is Paretian when the value of β is one, in which case this SWF will become the Sen SWF.[4] It is obvious from condition (9) that if only the richest person (or group) in society enjoy(s) the benefits of economic growth, the welfare of the society will not increase (as long as β<1). This SWF might be criticised for its bias in

[4] Also when β>1 the SWF will satisfy the condition of Paretianity. However, as our intention is to form a non-Paretian SWF we have concentrated in the range 0 to less than 1.

favour of the poor. If there is a rise in income of the poorest irrespective of the value of β and G (in the specified range, that is, between 0 and 1), social welfare must increase. Thus this SWF has some Rawlsian flavour. According to a Rawlsian SWF, however, if the richest person's income increases, social welfare remains unchanged. But in our SWF (with β<1), an increase in the income of the richest person (or group) leads to decrease in social welfare. This class of SWF (with β<1) is non-Rawlsian and not Paretian as well.

IV SWF decomposed by Components of income

The Gini coefficient can be decomposed by components of income as follows[5]:

(12) $$G = \sum_{i=1}^{k} S_i C_i$$

where $S_i = \frac{\mu_i}{\mu}$, the factor share of component i and C_i is the concentration coefficient of factor i. The concentration coefficient of the factor is calculated using the same formula as the Gini coefficient, only the ranking will remain the same as in the case of the Gini coefficient.[6] The value of the coefficient lies between (-1, 1) and, most importantly, it satisfies the Pigou-Dalton condition of transfer. The deviation of the Gini coefficient from the concentration coefficient (ie, C_i-G) indicates the direction of inequality augmenting or reducing effect of the component i. Clearly, if certain components of income accrue relatively more to poorer people (for example, government's cash transfer payments) the concentration coefficient will be negative. In contrast, if the component of income accrues more to the rich people (say, investment income) the concentration coefficient would be positive and will exceed the value of the Gini coefficient.

---

[5] See Rao (1967).

[6] When a specific factor income is arranged in ascending order of total income and the proportion of factor incomes are plotted against the proportions of income units, we get the concentration curve. One minus twice the area of the concentration curve is the concentration index. Unlike Lorenz curve, the concentration curve may lie above the 45^0 diagonal and in that case the concentration index will be negative.

Proposition

Total welfare is represented as a weighted sum of individual welfare of various factor components. Thus:

(13) $$W = \sum_{i=1}^{n} a_i W_i$$

where W_i is the welfare of the ith component and a_i is the weight attached to the individual component's welfare.

Theorem 2

The generalized SWF can be represented as the weighted sum of individual component's welfare.

Proof:

Using (12) and (13) it is easy to demonstrate the following:

(14) $$\begin{aligned} W &= \mu^{\beta}(1-G) \\ &= \sum_i [(\frac{\mu_i}{\mu})^{1-\beta}][\mu_i^{\beta}(1-C_i)] \\ &= \sum_i a_i W_i \end{aligned}$$

where $a_i = (\frac{\mu_i}{\mu})^{1-\beta}$, the weight attached to the ith factor component and $W_i = \mu_i^{\beta}(1-C_i)$ is the welfare due to the component i.

In this case, the relative welfare due to component i can be found as:

(15) $$\frac{a_i W_i}{W} = (\frac{\mu_i}{\mu})(\frac{1-C_i}{1-G})$$

The trade-off coefficient β does not appear here. The parameter β serves as a trade-off coefficient between equity and efficiency. When we are interested in estimating the relative contribution of one component to total welfare, the question of trade-off between efficiency and equity does not arise. The last term in the parenthesis on the right hand side of equation (15) ie, $\frac{1-C_i}{1-G}$ has an interesting interpretation and can be called 'relative equity of component *i*'. We have already mentioned that if the concentration coefficient of any component is higher (lower) than the overall Gini, the component has an inequality augmenting (reducing) effect. Thus, if the value of the relative equity is greater (less) than 1 the *i*th component will have an inequality reducing (augmenting) effect. This means that an extra dollar directed to this component will decrease society's inequality. Moreover, the relative welfare share due to component *i* depends on the relative mean income and the relative equity of component *i*.

The effects of economic growth in a component on the total welfare of the society can be obtained by finding the elasticity of total welfare with respect to the mean income of the *i*th component, as follows:

(16)
$$\begin{aligned}\eta_{\mu_i}^{W} &= \frac{\partial W}{\partial \mu_i}\frac{\mu_i}{W}\\ &= \frac{\mu_i}{\mu}\frac{1-C_i}{1-G} + (\frac{\mu_i}{\mu})(\beta - 1)\end{aligned}$$

This[7] is equal to the relative share of the component for β=1 (because the second term on the right hand side vanishes for β=1). When the SWF is non-Paretian, that means when β<1, the elasticity is less than the relative share (in this case the second term on the right-hand side is negative). If the factor share of the component is high, the second term will be large and will tend to reduce the elasticity more. Of course, it is also true that if the factor share of the component is small the reducing effect will be small. The decision-maker, comparing these elasticities of different components of income, might use his/her judgement for an equitable policy prescription. The effect of an increase in government cash benefits can be found using this procedure. Indeed, there have been attempts by government to make non-cash benefits, such as subsidised housing, education, medicare etc. The elasticity provides a clearer picture of the effects on total welfare of a proportionate change in any of the income components. Thus the policy maker will be better able to assess whether an increase in social insurance, for example, in terms of health protection to all in equal

[7] The formal proof is in the appendix.

proportion, or a proportionate increase in wage rate will be more effective from a social welfare point of view.

Lemma 1

For a non-Paretian SWF the effect of economic growth on a component (when there is no change in inequality) will be less than the share of welfare of the component.

The proof of lemma 1 follows from equations (15) and (16).

Since in most of the cases the legitimacy of efficiency criterion is very high, policy makers might consider a high value of β. For a high value of β the reducing effect of the second term on the right-hand side of equation (16) will be small. For those cases where there are components with high factor share along with high elasticity value, a cautious examination of the relative equity of the component is very important. A component with a high relative factor share can generate a high elasticity value. However, if it has a considerable inequality augmenting effect due to low relative equity (sufficiently less than one) that component should not be the target component.

V Conclusion

In all aspects of public policy there are both positive and normative issues that must be addressed. Welfare economics provides the necessary foundations upon which positive issues can be analyzed and therefore promotes a more informed debate of normative issues. Social decision makers have a set of value judgments that guide them in making policy decisions. For most social states the objective of the decision maker is to allocate the scarce resources and also to promote an equitable distribution of those resources. Thus ‘equity’ and ‘efficiency’ become two important arguments of the social welfare function. For several years, Pareto Optimality, as the definition of efficiency, has established a wide range of consensus. The notion of efficiency based on Pareto Optimality is not a positive concept. The Pareto optimality criterion is a value judgment, which intentionally avoids interpersonal comparison of utility. Pareto optimality rejects the ethics that individuals have equal capacity to enjoy a given share of income. It gives greater weights to the people having higher income. With this judgment a growth process accruing money only to the richest person of the society is desirable. Pareto optimality and also a Pareto utility function (since a Paretian utility function is a linear function of income) are indifferent with respect to the economic unit that receive an additive increase of income. The conclusion will be different if the increase is a percentage of the economic unit's income. Since the Pareto principle avoids interpersonal comparison of utility, hence, is utilitarian, and the utility function is concave (linear for Pareto), an increase in income of any economic unit will increase both individual and total welfare. By viewing the real world, where for most social situations the majority of the population is poor compared to a very small number of rich people, it is difficult to accept that the majority would want social policies to be formulated on Pareto criteria. Thus for a decision maker a policy is not necessarily welfare-augmenting simply because it raises efficiency. The flexibility of the SWF and the trade-off between equity and efficiency is another aspect to consider. This paper has argued that it is possible to generalize the widely used Sen SWF, which can be non-Paretian under special circumstances.

Also it has been demonstrated that a disaggregation of this SWF by components of income can be done. It is shown that the relative contribution of welfare of one factor depends on two components: the factor share and the relative equity of the component. It is also shown that the growth of one component can generate a negative element when the SWF is non-Paretian. The value of this negative effect is larger, the larger the factor share of the component. The method is important in benefit/cost analysis and in the determination of the target component.

Appendix

Derivation of equation (16)

$$W = \mu^{\beta} - \mu^{\beta} G$$
$$= (\sum \frac{n_i}{n}\mu_i)^{\beta} - (\sum \frac{n_i}{n}\mu_i)^{\beta-1}(\sum \frac{n_i}{n}\mu_i C_i)$$

and

$\mu = \sum \mu_i$ and $\mu G = \sum \mu_i C_i$. Hence, we can derive

$$\frac{\partial W}{\partial \mu_i} = \beta(\sum \mu_i)^{\beta-1}\frac{n_i}{n} - (\beta-1)(\sum \mu_i)^{\beta-2}\sum \mu_i C_i - (\sum \mu_i)^{\beta-1} C_i$$
$$= \beta\mu^{\beta-1} - (\beta-1)\mu^{\beta-2}\mu G - \mu^{\beta-1} C_i$$
$$= [\beta\mu^{\beta-1} - (\beta-1)\mu^{\beta-1} G - \mu^{\beta-1} C_i]$$

Thus the elasticity can be written as

$$\eta^{W}_{\mu_i} = \frac{\partial W}{\partial \mu_i}\frac{\mu_i}{W}$$
$$= \frac{[\beta\mu^{\beta-1} - \mu^{\beta-1}(\beta-1)G - \mu^{\beta-1}C_i]\mu_i}{\mu^{\beta}(1-G)}$$
$$= \frac{[\beta - (\beta-1)G - C_i]\mu_i}{\mu(1-G)}$$
$$= \frac{[\beta - (\beta-1)G - C_i - 1 + 1]\mu_i}{\mu(1-G)}$$
$$= \frac{[(\beta-1) - (\beta-1)G]\mu_i + (1-C_i)\mu_i}{\mu(1-G)}$$

that is,

$$\frac{\mu_i(\beta-1)[1-G]}{\mu(1-G)} + \frac{\mu_i(1-C_i)}{\mu(1-G)}$$
$$= \frac{\mu_i(\beta-1)(1-G)}{\mu(1-G)} + \frac{\mu_i(1-C_i)}{\mu(1-G)}$$
$$= \frac{\mu_i}{\mu}(\beta-1) + \frac{\mu_i(1-C_i)}{\mu(1-G)}$$

References

Dagum, C. (1990): 'Relationship between income inequality measures and social welfare functions', *Journal of Econometrics*, 43, 91-102.

Dagum, C. (1993): 'The social welfare cases of Gini and other inequality meaures', *Statistica*, 53, 3-30.

Kondor, Y. (1975): “Value judgments implied by the use of various measures of income inequality”, *Review of Income and Wealth*, 21, 309-21

Rao, V. M. (1967): ‘Two decomposition of concentration ratio’, *Journal of Royal Statistical Society*, Series A, 132, 428-435

Saunders, P. (1994): *Welfare and inequality*, Cambridge University Press, Cambridge.

Sen, A. K. (1974): ‘Informational bases of alternative welfare approaches: Aggregation of income distribution’, *Journal of Public Economics*, 3, 387-403.

Sen, A. K. (1976): ‘Real national income’, *The Review of Economic Studies*, 43, 19-39.

Sheshinski, E. (1972): ‘Relation between a social welfare function and the Gini index of inequality’, *Journal of Economic Theory*, 4, 98-100.

Yitzhaki, S. (1979): ‘Relative deprivation and the Gini coefficient’, *Quarterly Journal of Economics*, 93, 321-24.

Yitzhaki, S. (1982): ‘Relative deprivation and the economic welfare’, *European Economic Review*, 17, 19-113.