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The Solow Growth Model with Keynesian Involuntary Unemployment

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Non-technical Summary

Unemployment, which is undoubtedly a fundamental macroeconomic issue, is treated as a short term phenomenon affecting fluctuations, but, surprisingly, is completely neglected in Neoclassical growth models. It is also surprising that a fiscal policy or, more generally, a shock which affects one of the components of the aggregate demand, produces completely different effects on GDP and employment using Neoclassical supply-driven models or Keynesian demand-driven models. The different vision of the functioning of the economy is translated to the disagreement concerning the implementation of austerity policies in order to face the current double problem of high public debts and low economic growth.

The aim of this paper is to extend the Solow growth model by taking into account for the Keynesian unemployment, i.e. unemployment due to the weakness of aggregate demand. In order to introduce a Keynesian unemployment in the Solow model, we relax the hypothesis that investments are determined by aggregate savings in order to achieve full-employment. More precisely, the only difference with respect to the standard Solow model is that we introduce in our model one supplementary equation, the investment function, and one supplementary variable, the unemployment rate. In particular, using a very simple investment function, we show that the instantaneous equilibrium may be characterized by the presence of involuntary unemployment if investments are below a threshold value. We also show that, for an under-capitalized economy, the capital per unit of effective labor and the unemployment rate increase over time during the transition phase, until the economy reaches its steady state which may be characterized by a positive value of the unemployment rate.

Then, we show that an increase in the saving rate produces a negative effect on unemployment and on GDP, both in the short and in the long run. This result is due to the fact that our base model, even if it presents many features of Neoclassical models (i.e. the production function allows for factor substitutability, the representative firm maximizes its profit, factors are remunerated at their marginal productivity, and prices are perfectly flexible), in reality it works as a Keynesian model, i.e. it is demand driven. Thus, in the base model, an increase in the saving rate provokes a reduction in private consumption and in the aggregate demand and then increases unemployment.

Then, we modify the investment function in a way which allows us to take into account for the crowding-

in/crowding-out effect on investments. In particular, we introduce a parameter that measures the degree of the crowding-in/crowding-out effect, i.e. the fact that a change in one of the components of the aggregate demand affects investments. We show that if this parameter is equal to zero, the model coincides with our base model, i.e. the Keynesian demand-driven model. If the parameter is equal to one, the model coincides with the Solow model and the unemployment rate remains unchanged. Finally, and more interestingly, if the parameter lies between zero and one, the model becomes an intermediate model which allows to take into account that a shock or a policy which increases the aggregate demand (for example a reduction in the saving rate or the implementation of an expansionary fiscal policy) stimulates GDP and reduces unemployment (while, in Neoclassical models, the real effect is nil), but, at the same time, produces a (partial) crowding-out effect on investments (that is not taken into account in Keynesian models).

ABSTRACT

The aim of this paper is to extend the Solow model in a way that permits to endogenize unemployment. Starting from a Neoclassical growth model, as the Solow model, we introduce a mechanism that allows us to determine the Keynesian unemployment, i.e. unemployment that is caused by the weakness of the aggregate demand. Using our base model, that works as a Keynesian demand-driven model, we find that an increase in the aggregate demand (due to a reduction in the saving rate or to an increase in public expenditures) reduces unemployment and stimulates the GDP. Then, we modify the investment function in order to take into account for the crowding-in/crowding-out effect on investments. This allows us to build a model which is between Neoclassical supply-driven models and Keynesian demand-driven models.

JEL Classification: O40; E13; E12; J60

Keywords: Growth models; Neoclassical models; Keynesian models; Involuntary unemployment.

The Solow Growth Model with Keynesian Involuntary Unemployment

Riccardo Magnani

Resume non technique

Le chômage, qui est sans nul doute un sujet fondamental en macro-économie, est traité comme un phénomène de court terme et, de façon surprenante, est complètement négligé dans les modèles néoclassiques de croissance. Il est également étonnant de constater que la politique budgétaire ou, plus généralement, un choc affectant l'une des composantes de la demande agrégée produisent des effets complètement différents sur le PIB et l'emploi selon qu'on se situe dans un cadre néoclassique ou keynésien. Cette vision différente du fonctionnement de l'économie explique le désaccord qui existe aujourd'hui entre les économistes quant à l'efficacité à attendre des politiques d'austérité ou de relance pour faire face au double problème posé par les niveaux élevés des dettes publiques et par la faiblesse de la croissance.

Le but de ce papier est de proposer une extension du modèle de croissance de Solow en prenant en compte le chômage keynésien, c'est-à-dire le chômage provoqué par la faiblesse de la demande agrégée. Afin d'introduire le chômage keynésien, nous relâchons l'hypothèse du modèle de Solow selon laquelle l'investissement est déterminé par l'épargne agrégée de façon à garantir le plein-emploi. Plus précisément, nous introduisons dans un modèle de Solow standard une équation supplémentaire, la fonction d'investissement, et une variable supplémentaire, le taux de chômage. Avec une fonction d'investissement très simple, nous montrons en particulier que l'équilibre instantané de l'économie peut être caractérisé par la présence de chômage involontaire lorsque les investissements sont en dessous d'une valeur seuil. Nous montrons aussi que, pour une économie sous-capitalisée, le capital par unité de travail et le taux de chômage augmentent au cours de la phase de transition, jusqu'au moment où l'économie atteint son état d'équilibre qui peut être caractérisé par une valeur positive du taux de chômage.

Nous montrons également que l'augmentation du taux d'épargne produit un effet négatif sur le chômage et le PIB, à la fois à court et à long terme. Ce résultat est dû au fait que notre modèle de base, même s'il présente de nombreuses caractéristiques des modèles néoclassiques (la fonction de production est à facteurs substituables, l'entreprise représentative maximise son profit, les facteurs sont rémunérés à leur productivité marginale et les prix sont parfaitement flexibles), fonctionne en réalité comme un modèle keynésien, c'est à dire que la production est déterminée par la demande. Ainsi, une augmentation du taux d'épargne provoque une réduction de la consommation privée et de la demande agrégée, ce qui affecte négativement le chômage.

Nous modifions alors la fonction d'investissement de façon à prendre en considération l'ampleur de l'effet d'éviction, c'est à dire la mesure dans laquelle un changement dans l'une des composantes de la demande agrégée affecte les investissements. Pour cela, nous introduisons un paramètre mesurant le degré d'éviction. Si ce paramètre est égal à zéro, le modèle coïncide avec notre modèle de base, à savoir un modèle keynésien où la production est déterminée par la demande. Si le paramètre est égal à un, le modèle coïncide avec le modèle de Solow et le taux de chômage reste inchangé. Enfin, si le paramètre est compris entre zéro et un, le modèle devient un modèle intermédiaire qui permet à la fois de prendre en compte le fait qu'un choc ou une stimulation de la demande agrégée (par exemple une réduction du taux d'épargne ou une politique budgétaire expansionniste) provoquent un effet positif sur le PIB et réduisent le chômage (dans un modèle néoclassique, l'effet réel serait nul), mais qu'ils produisent en même temps un effet d'éviction (partielle) sur les investissements (effet qui n'existerait pas dans un modèle keynésien).

RESUME COURT

Le but de ce papier est de proposer une extension du modèle de croissance de Solow qui permet d'endogénéiser le chômage. A partir du modèle de Solow, nous introduisons un mécanisme qui nous permet de déterminer le chômage keynésien, c'est-à-dire le chômage provoqué par la faiblesse de la demande agrégée. Notre modèle de base fonctionne alors comme un modèle keynésien où la production est déterminée par la demande : l'augmentation de la demande agrégée (due à une réduction du taux d'épargne ou à une augmentation des dépenses publiques) réduit le chômage et stimule le PIB. Ensuite, nous modifions la fonction d'investissement, afin de tenir compte de l'effet d'éviction sur les investissements. Cela nous permet de construire un modèle qui se situe entre les modèles néoclassiques où la production est déterminée par l'offre des facteurs et les modèles keynésiens où la production est déterminée par la demande.

Classification JEL : F40 ; C63 ; C68

Mots clés : Modèles de croissance ; Modèles Néoclassiques ; Modèles Keynésiens ; Chômage involontaire.

THE SOLOW GROWTH MODEL WITH KEYNESIAN INVOLUNTARY UNEMPLOYMENT

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1. INTRODUCTION AND MOTIVATION

It is quite surprising that Neoclassical growth models have completely neglected a fundamental macroeconomic issue such as unemployment. Unemployment is treated as a short term phenomenon affecting fluctuations, but not as a long term issue. In contrast, empirical data (for example for the US) show that not only GDP growth rates but also unemployment rates fluctuate around a trend that would deserve to be taken into account in growth models.

It is also surprising that a macroeconomic shock such a change in public expenditures or, more generally, in one of the components of the aggregate demand, produces completely different effects whether using a Neoclassical supply-driven model or a Keynesian demand-driven model. In particular, the different vision of the functioning of the economy reflects the disagreement concerning the implementation of austerity policies in order to face the current double problem of high public debts and low economic growth.

The aim of this paper is to propose a growth model which extends the standard Solow model by taking into account for the unemployment. Starting from the Solow model, we introduce a mechanism that allows us to endogenize the unemployment rate. In particular, the kind of unemployment considered in our paper is the Keynesian unemployment. For Keynesian unemployment we mean the unemployment that is caused by the weakness of the aggregate demand.

In order to introduce a Keynesian unemployment in the Solow model it is necessary to relax the fundamental hypothesis made in the Classical and Neoclassical theory, i.e. the full-utilization of the production factors. Very few works have considered the possibility of under-utilization of production factors in a Neoclassical framework. Concerning the utilization of capital, Chatterjee (2005) examines the implications of capital utilization for the dynamics of economic growth and convergence. Starting from the evidence that in the economies there is less than full utilization of capital and that the depreciation rate is not constant, he shows that endogenizing the rate of capital utilization can lead to empirically plausible speeds of convergence in economic growth models. Concerning the utilization of labor, the only paper in the literature that in our knowledge analyzed a Solow model with endogenous Keynesian unemployment is by Backhouse (1981). He presented a model where the long-run equilibrium growth path is the "Solow

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steady state" and where the short-run equilibrium, described by the IS-LM curves, permits the presence of unemployment. According to the author, the short-run equilibrium, defined "in the Keynesian manner" by assuming that the nominal wage and the capital stock are constant, is characterized by two types of disequilibrium that differentiate the Keynesian model from the Neoclassical model: the presence of unemployment (caused by wage stickiness) and the fact that the rate of interest is different from the marginal product of capital (since the capital stock is fixed in the short run). Then, the author shows that the economy converges towards a steady state characterized by full-employment.

We do not agree with the view that Keynesian unemployment is caused to wage rigidity. Even if one of the Keynesian arguments is that flexible money wages produces destabilizing effects in the economy,[1] it is wrong to argue that in the Keynesian view unemployment is caused by wage rigidity. In fact, if unemployment was provoked by wage rigidity, full-employment could be achieved by reducing the wage level. But this is exactly the contrary of the Keynesian view: a reduction in the wage level reduces households' income, contracts consumption, and produces negative effects on the real activity and employment. Of course, wage rigidity is one of the causes of unemployment but, in the Keynesian view, the key element explaining unemployment is the weakness of the aggregate demand and not wage rigidity.

In our paper, we first discuss in Section 2 the characteristics of the labor market and of the instantaneous equilibrium in the presence of Keynesian unemployment, i.e. in the case in which the weakness of the aggregate demand provokes involuntary unemployment. In particular, we compare our view of the labor market with the one of Patinkin (1965) and of the disequilibrium theory (Benassy, 1975, Malinvaud, 1977, and Barro and Grossman, 1971). In Section 3, we present our base model which extends the Solow model in order to endogenize the unemployment rate. More precisely, we relaxe the hypothesis that investments are determined by aggregate savings in order to achieve full-employment. Thus, the only difference with respect to the standard Solow model is that we introduce one supplementary equation, i.e. the investment function, and one supplementary variable, i.e. the unemployment rate. In particular, using a very simple investment function, we show that the instantaneous equilibrium may be characterized by the presence of involuntary unemployment if investments are below a threshold value. Concerning the dynamics, we show that an under-developed economy converges towards its steady-state equilibrium that may be characterized by a positive value of the unemployment rate and, during the transition phase, the capital per unit of effective labor increases and the unemployment rate increases over time, until the economy reaches its steady state. Then, we show that an increase in the saving rate produces a negative effect on employment and on GDP, both in the short and in the long run. This result is due to the fact that our base model, even if it presents many features of Neoclassical models (i.e. the production function allows for

[1] Keynes observed that a policy of flexible money wages "would be to cause a great instability of prices, so violent perhaps as to make business calculations futile in an economic society functioning after the manner of that in which we live. To suppose that a flexible wage policy is a right and proper adjunct of a system which on the whole is one of laissez-faire, is the opposite of the truth" (1936, p. 269).

factor substitutability, the representative firm maximizes its profit, factors are remunerated at their marginal productivity, and prices are perfectly flexible), in reality it works as a Keynesian model, i.e. it is demand driven. Thus, in the base model, an increase in the saving rate provokes a reduction in private consumption and in the aggregate demand and then increases unemployment. In Section 4, we modify the investment function in a way which allows us to take into account for the crowding-in/crowding-out effect on investments. In particular, we introduce a parameter that measures the degree of the crowding-in/crowding-out effect, i.e. the fact that a change in one of the components of the aggregate demand affects investments. We show that if this parameter is equal to zero, the model coincides with our base model, i.e. the Keynesian demand-driven model. If parameter is equal to one, the model coincides with the Solow model and the unemployment rate remains unchanged. Finally, and more interestingly, if the parameter lies between zero and one, the model becomes an intermediate model which allows to take into account that a shock or a policy which increases the aggregate demand (for example a reduction in the saving rate or the implementation of an expansionary fiscal policy, as analyzed in Section 5) stimulates GDP and reduces unemployment (while, in Neoclassical models, the real effect is nil), but, at the same time, produces a (partial) crowding-out effect on investments (that is not taken into account in Keynesian models).

2. The labor market and the instantaneous equilibrium

In the Solow model, the representative firm demands the optimal quantity of labor and capital in order to maximize its profit given a technological constraint. At the optimum, the marginal productivity of each factor coincides with their marginal cost. Price flexibility permits to equilibrate factor demands with factor supplies. The remuneration of production factors are then determined such that the representative firm completely uses the production factors available in the economy. At each period, total production is then fixed at the level corresponding to the full-employment of production factors. This implies that, at each period, the sum of consumption, investments, public expenditures and net exports is also fixed at a predetermined level. In particular, in the Solow model which considers a closed economy without the government, consumption is determined by a fraction of real incomes (corresponding to full-employment) while investments, which are not linked to an optimal behavior of the representative firm, are determined in a residual way. This implies that the macroeconomic equilibrium condition, which states that investments equal aggregate savings, determines the level of investments. Investments are then saving-driven. The key hypothesis of the Solow model is that investments adjust in order to be equal to the aggregate savings corresponding to full-employment. In a Keynesian model, instead, each component of the aggregate demand is determined by a specific equation, implying that the sum of the components of the aggregate demand determines real GDP. In particular, if investments are lower than a threshold level (for example, due to investors' pessimism), then full-employment cannot be achieved and unemployment, due to the weakness of the aggregate demand, appears. Consequently, in a Keynesian model, the macroeconomic equilibrium condition between investments and aggregate savings determines the level of real

GDP. In other words, the introduction of a macroeconomic investment function, which is not directly related to the optimal behavior of the representative firm, implies that the competitive equilibrium may be characterized by the presence of unemployment.

Consider now the labor market. Patinkin (1965) asserted that the equilibrium real wage rate is the level that equates the amounts demanded and supplied of labor, i.e. the level that guarantees the labor market clearing (1965, p. 203). Given that in the Keynesian theory the presence of involuntary unemployment implies that the labor market is not cleared, Patinkin states that "Keynesian economics is the economics of unemployment disequilibrium" (1965, pp. 337-338). Patinkin (1965) and Barro and Grossman (1971) analyzed, in this context of general disequilibrium, the effect of a reduction in aggregate demand.[2] This induces a reduction in employment and, in the case in which the wage rate remains unchanged, the quantity of labor demanded becomes lower than the full-employment level. However, in their analysis, the quantity of labor demanded does not belong to the marginal productivity of labor curve. In particular, this off-demand-curve analysis defended by Patinkin (1965) and Barro and Grossman (1971) implies that, for a given level of employment lower than full-employment, the real wage is lower than the marginal productivity of labor, which is completely incoherent with the firm's profit maximization. It is interesting to note that even Keynes asserted that in a competitive economy the real wage is equal to the marginal product of labor (1936, pp. 5 and 17).

The Patinkin's view of Keynesian unemployment has been widely criticized by Davidson (1967 and 1983). According to Davidson, the marginal productivity of labor used by Patinkin and, in general, by Neoclassical economists does not represent the labor demand curve in an economy based on Keynes's theory. In particular, Davidson states that the intersection of the aggregate supply and demand curves determines the equilibrium level of employment and the marginal productivity of labor has to be interpreted as a market equilibrium curve which determines, for a given the level of employment, the equilibrium real wage.

Following Davidson (1967 and 1983), once the production level is determined by the aggregate demand, the level of employment is given by the inverse of the production function, $L^d = F^{-1}(Y)$, i.e. employment represents the quantity of labor necessary to produce Y. Given the number of individuals employed, it is possible to determine the unemployment rate: $u = \frac{L^s - L^d}{L^s}$, where the labor supply L^s can be considered endogenous or exogenous as in the Solow model. The unemployment rate can be interpreted as the *equilibrium unemployment rate*[3] in the sense that it is the *only* level of the unemployment rate that guarantees the macroeconomic equilibrium between investments and aggregate savings or, equivalently, the equilibrium in the market of goods. Figure 1 shows what happens in the labor market if the level of production, Y_1, is lower

[2] In particular, Barro and Grossman (1971) assumed that the reduction in aggregate demand is due to a high price level while, as have we have already said, Keynesian theory states that unemployment is not caused by price rigidity.

[3] It is important to highlight that the notion of equilibrium unemployment rate used in our paper is completely different with respect to the notion used in search and matching models in which the equilibrium unemployment rate is the rate such that the number of people finding a job is equal to the number of people who lose a job.

Figure 1 – Involuntary unemployment in the labor market (first interpretation)

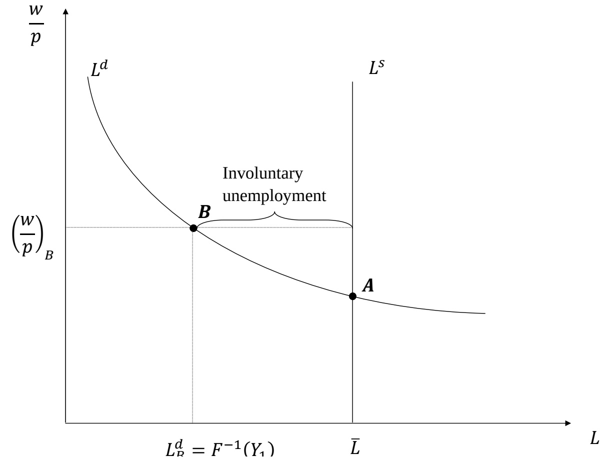
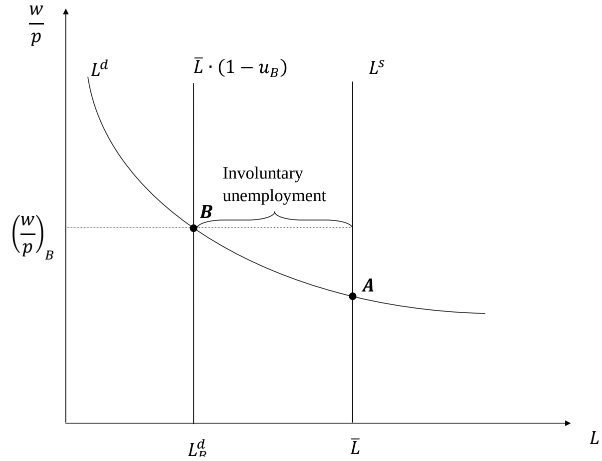


Figure 2 – Involuntary unemployment in the labor market (second interpretation)



than the full-employment level. The level of employment, L_B^d, is determined by the inverse of the production function corresponding to the production level Y_1. The real wage rate is determined along with the labor demand curve, and the difference between labor supply, supposed to be exogenous and equal to $\bar{L}$, and labor demand determines involuntary unemployment.

The mechanisms of the labor market that we have described are essentially equivalent to that discussed by Davidson (1967 and 1983), i.e. the aggregate demand determines the level of production which in turn determines the level of employment, while the marginal productivity of labor determines the level of the real wage. However, we think that the fact that the wage rate is determined by the level of the marginal productivity of labor is not completely satisfactory in order to explain the functioning of the labor market. In fact, the equality between the marginal productivity of labor and the real wage indicates that, in order to maximize profits, the quantity of labor demanded by the representative firm must be such that the marginal productivity of labor coincides with the real wage. Thus, this equality cannot determine the real wage. In addition, if the quantity of labor demanded has already been determined (given that the production level is determined by the aggregate demand), then firms have nothing to maximize implying that the first order condition for profit maximization is useless.

An alternative way to analyze the labor market, presented in Figure 2, consists to consider that the macroeconomic condition determines the equilibrium unemployment rate, u_B. Then, it is possible to plot the (vertical) curve representing the total quantity of labor supplied, $\bar{L}\cdot(1-u_B)$. The profit-maximization condition determines the labor demand function, $L^d = f\left(\frac{w}{p}\right)$, as in standard Neoclassical models. The intersection between the labor demand curve and the vertical curve representing the *effective* quantity of labor supplied determines the quantity of labor employed, $L_B^d = \bar{L}\cdot(1-u_B)$, and the "equilibrium" wage rate $\left(\frac{w}{p}\right)_B$. Finally, the production function determines the level of production level depending on the quantity of labor employed, $Y_1 = F(L_B^d, \overline{K})$.

A very important aspect to note is that point A in both figures, i.e. the intersection between labor demand and labor supply, does not represent an equilibrium in the case in which the aggregate demand (and then the production level) is equal to Y_1, i.e. lower than the full-employment level. In fact, at point A, aggregate saving is greater than investments or, equivalently, production is greater than aggregate demand. In contrast, point B in both figures represents the equilibrium of the economy in the case in which aggregate demand (and then the production level) is equal to $Y_1 < \bar{Y}$. This equilibrium can be defined as an *under-employment equilibrium*, in the sense that the weakness of the aggregate demand provokes involuntary unemployment. Nevertheless, it is an equilibrium: the market of goods and services is in equilibrium since the production is equal to the aggregate demand, and the labor market is in equilibrium since the demand of labor is equal to the effective quantity of labor supplied.

2.1. Under-unemployment equilibrium and factor prices

Figures 1 and 2 in the previous section have illustrated that the wage rate is greater in a situation of under-employment equilibrium than in a situation of full-employment equilibrium. In other words, starting from full-employment, a reduction in the aggregate demand provokes involuntary unemployment and increases the level of the wage rate. The wage rate is then counter-cyclical. The same result can be obtained by considering the isoquants of production (see Figure 3). Consider, first, a full-employment situation in which the production level is $\overline{Y}$ and the disposable capital ($\overline{K}$) and labor ($\overline{L}$) are completely used. In this case, the equilibrium prices of the production factors are given by the slope of the line tangent to the isoquant at point A. Then, consider a reduction in the aggregate demand which reduces production ($Y_1 < \overline{Y}$) and provokes involuntary unemployment, with the unemployment rate equal to u_B. By supposing that capital is still full-employed, the equilibrium prices of the production factors are given by the slope of the line tangent to the new isoquant at point B. The new slope is clearly greater, in absolute value, with respect to the previous one, implying that, with respect to the initial situation of full-employment, the wage rate is greater and the rate of remuneration of capital is lower. This result, which is perfectly analogous to that presented in the previous section, is due to the fact that capital is supposed to be fully-employed. It is then interesting to analyze the effect on the equilibrium prices of the production factors in the case in which the hypothesis of full-employment of capital is relaxed.

Suppose that the production level Y_1 is realized with full-employment of labor and under-employment of capital. The economy produces Y_1 by using $\overline{L}$ and $z_C \cdot \overline{K}$, where z represents the utilization rate of capital. In this case, the equilibrium prices of the production factors are given by the slope of the line tangent to the new isoquant at point C, implying that, with respect to the initial situation of full-employment, the wage rate is lower and the rate of remuneration of capital is greater. In this case, the wage rate is pro-cyclical.

Points B and C represent two extreme cases in which the reduction in production is obtained by

Figure 3 – Equilibrium prices of production factors

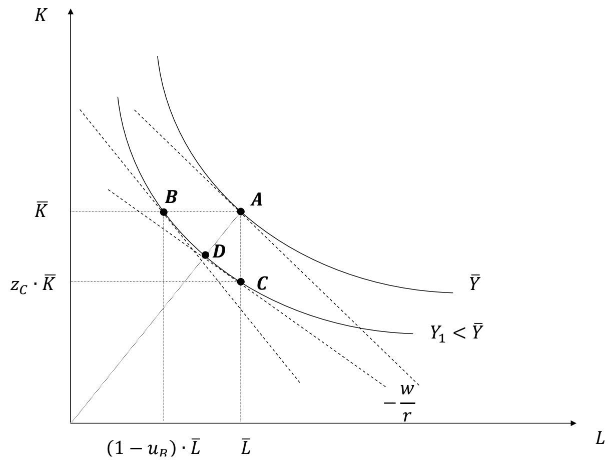
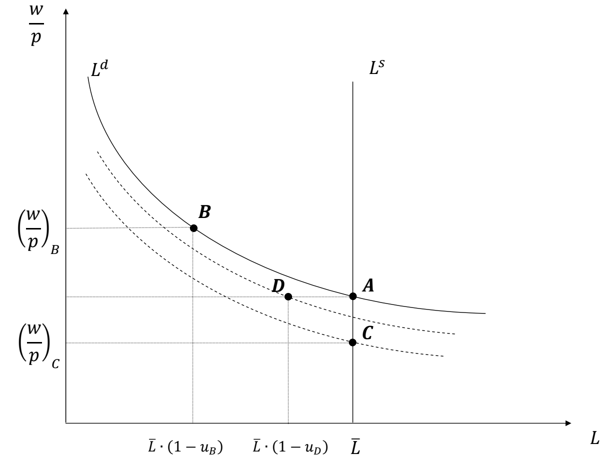


Figure 4 – The wage rate if both labor and capital are under-employed



reducing only one factor, while the other factor is assumed to be fully-employed. Of course, the production level Y_1 can be realized by assuming under-employment of both labor and capital, implying that the combination of labor and capital, which permits to produce Y_1, lies between points B and C. In particular, in point D, the wage rate and the rate of remuneration of capital are exactly the same as in the initial situation of full-employment. This implies that if the combination of labor and capital, permitting to produce Y_1, is chosen between points B and D, then the wage rate is counter-cyclical; in contrast, if it is chosen between the points D and C, then the wage rate is pro-cyclical.

Coming back to the analysis of the labor market, Figure 4, which is analogous to Figure 3, shows that starting from the full-employment equilibrium (point A), a reduction in the aggregate demand may produce (i) an increase in the wage rate (point B) if capital is assumed to be full-employed, or (ii) a decrease in the wage rate (point C) if labor is assumed to be fully-employed and the utilization rate of capital decreases (which implies that the marginal productivity of labor decreases and that the labor demand curve shifts downwards), or (iii) any intermediate case if both labor and capital are under-employed. The wage rate will range from $(w/p)_B$ to $(w/p)_C$. For example, in point D, in which both labor and capital are under-employed, the wage rate is exactly the same as in the initial situation of full-employment.

To conclude the analysis of the labor market, the weakness of aggregate demand may provoke the under-employment of labor, or capital, or both. Of course, the macroeconomic equilibrium condition can determine just one variable, i.e. the level of under-employment of one factor. Thus, to determine the level of under-employment of the other factor it is necessary to introduce an additional equation. Here, there are two possibilities. The first one is to introduce an equation describing the relationship, which can be econometrically estimated, between the unemployment rate and the utilization rate of capital. The second one is to introduce an equation which implies the hypothesis that the wage rate (which, as we have already said, must lie between two values) is determined by a bargaining process. In our paper, we do not treat these aspects and we simply assume that capital is fully-employed which implies that the utilization rate of capital is fixed to unit or that the wage rate is determined by negotiation at the highest level.

2.2. The Walras Law

Our interpretation of the functioning of the labor market implies that in order to take into account for Keynesian unemployment it is not necessary to introduce nominal nor real rigidities. For this reason, we assume, as in the Solow model, that all the prices are perfectly flexible implying that money is completely neutral. Finally, as in the Solow model, the economy is constituted by three markets, i.e. the market of goods, of labor and of capital, and the good produced in the economy is chosen as the *numéraire*.

The literature of disequilibrium proposed by Benassy (1975) and Malinvaud (1977) states that in the case of price rigidity and if only money can be used for transactions, demands are constrained and the Walras Law may be violated. However, price flexibility allows to restore the equilibrium. In our view of the functioning of the labor market, the situation is completely different. Price flexibility is not sufficient to restore equilibrium and the fact that the labor market is not cleared is not due to price rigidity but to the weakness of the aggregate demand.

Apparently, in a Keynesian economy characterized by the presence of involuntary unemployment, all markets are in equilibrium with the exception of the labor market, implying that, apparently, the Walras Law is violated. According to Clower (1965) and Leijonhufvud (1968), the Walras Law is not relevant for modeling situations characterized by the presence of involuntary unemployment, i.e. when the levels of income and production are endogenous. In particular, they state that an excess of supply in one market must be matched by an excess of demand in at least one other market, but only in terms of planned (or notional) demands and supplies, and not in terms of actual (or effective) demands and supplies.[4]

Here, we define the *effective* labor supply as the quantity of labor that people offer (at a given wage rate) subject to the constraint (that is exogenous for the individuals) represented by unemployment. We define the *effective* demand of goods as the quantity demanded depending on incomes related to the effective labor supply. First, the macroeconomic equilibrium condition between investments and aggregate savings determines the equilibrium unemployment rate and then the effective labor supply $(1-u)\cdot L^s$. Then, the Walras Law can be written as follows:

$$p\cdot\left[Y^s-Y^d\right]+w\cdot\left[(1-u)\cdot L^s-L^d\right]+r\cdot\left[K^s-K^d\right]=0 \qquad (1)$$

[4]In fact, with involuntary unemployment, planned labor supply is greater than labor demand, and planned demand for goods is greater than supply of goods, implying that Walras Law hold. At the same time, actual (or effective) labor supply is greater than labor demand, and actual demand for goods coincides with supply of goods. Walras Law does not hold.

In particular, by supposing that a fraction $1-s$ of the real income is consumed and by noting by I investments, the effective demand of goods is given by

$$Y^d = (1-s)\cdot\left[\frac{w}{p}\cdot(1-u)\cdot L^s + \frac{r}{p}\cdot K^s\right] + I.$$

In contrast, the following Walras Law is violated:

$$p\cdot\left[Y^s - Y^d\right] + w\cdot\left[L^s - L^d\right] + r\cdot\left[K^s - K^d\right] \neq 0 \tag{2}$$

In fact, if $L^s = L^d$ and $K^s = K^d$, then $(1-s)\cdot\left(\frac{w}{p}\cdot L^s + \frac{r}{p}\cdot K^s\right) + I$ represents the notional demand Y^d, that is different from Y^s.

Walras Law defined in equation 1 always holds, with equilibrium prices or with disequilibrium prices. One of the three equilibrium conditions, i.e. $Y^s = Y^d$, or $(1-u)\cdot L^s = L^d$, or $K^s = K^d$, is redundant, and the good produced in the economy can be used as *numéraire*, implying $p = 1$. An interesting case of disequilibrium prices is the case in which the real wage is fixed at the (disequilibrium) level that permits to achieve the full-employment $L^d = L^s$. In this case, the real wage is then fixed at a level lower than the equilibrium level. Then, there is an excess demand of labor, i.e. $L^d > (1-u)\cdot L^s$, and an excess supply of goods, i.e. $Y^s > Y^d$.

3. The Solow model with endogenous unemployment

Here we present our base model which extends the Solow model by introducing Keynesian unemployment and by supposing that capital is fully-employed. From one hand, our model is a Neoclassical model in the sense that the production function allows for factor substitutability, the representative firm maximizes its profit, factors are remunerated at their marginal productivity, and all prices are perfectly flexible.[5] On the other hand, the model works as a Keynesian model. Even if the money market is not taken into account, our model is demand-driven and, in particular, if the level of aggregate demand is low, unemployment appears.

As in the Solow model, the production function is a Cobb-Douglas function with labor-augmenting productivity:

$$Y(t) = K(t)^{\alpha}\cdot\left[A(t)\cdot L(t)\cdot(1-u(t))\right]^{1-\alpha} \tag{3}$$

where $K(t)$ represents the capital stock, $A(t)$ represents the productivity level assumed to grow at a constant rate g_A, $L(t)$ represents the working-age population assumed to grow at a constant

[5]Given that prices are assumed to be perfectly flexible, money is completely neutral. It is then useless to introduce in our model the Keynesian LM curve $\frac{\overline{M}}{p} = \frac{M^d}{p}(r,Y)$. This equation would determine the price level implying that a change in money supply $\overline{M}$ provokes a proportional change in p, no change in relative prices and then no real effects.

rate n, and $u(t)$ represents the unemployment rate. $L(t) \cdot (1-u(t))$ represents the number of workers[6] and $A(t) \cdot L(t) \cdot (1-u)$ represents the number of units of effective labor. The initial level of productivity and of the working-age population are normalized to 1, then: $A(t) = e^{g_A t}$ and $L(t) = e^{nt}$. Finally, we define $A(t) \cdot L(t)$ as the number of units of effective *potential* labor, in the sense that it represents the number of units of effective labor in the case full-employment, $u(t) = 0$.

Before presenting the resolution of the model, it is important to detail the notation used:

- The capital per unit of effective potential labor is defined as:

$$\hat{k}(t) = \frac{K(t)}{A(t) \cdot L(t)}$$

- The capital per unit of effective labor is defined as:

$$\tilde{k}(t) = \frac{K(t)}{A(t) \cdot L(t) \cdot (1-u(t))} = \frac{\hat{k}(t)}{1-u(t)}$$

- Real GDP is then given by:

$$\begin{aligned} Y(t) &= \left[\frac{K(t)}{A(t) \cdot L(t) \cdot (1-u(t))}\right]^{\alpha} \cdot A(t) \cdot L(t) \cdot (1-u(t)) = \tilde{k}(t)^{\alpha} \cdot A(t) \cdot L(t) \cdot (1-u(t)) \\ &= \left[\frac{K(t)}{A(t) \cdot L(t)}\right]^{\alpha} \cdot A(t) \cdot L(t) \cdot (1-u(t))^{1-\alpha} = \hat{k}(t)^{\alpha} \cdot A(t) \cdot L(t) \cdot (1-u(t))^{1-\alpha} \end{aligned}$$

- Real GDP per unit of effective potential labor is given by:

$$\hat{y}(t) = \frac{Y(t)}{A(t) \cdot L(t)} = \hat{k}(t)^{\alpha} \cdot (1-u(t))^{1-\alpha}$$

3.1. Instantaneous equilibrium

The macroeconomic equilibrium condition states that investments are equal to aggregate savings. In a closed economy without government and by assuming that the representative agent saves a constant fraction s of his revenue Y, the macroeconomic equilibrium condition is:

$$I(t) = s \cdot Y(t)$$

The key assumption of our model is that investments are not determined by the macroeconomic equilibrium condition, i.e. investments are not saving-driven, but they are determined

[6] Note that even in standard Neoclassical models $L(t) \cdot (1-u(t))$ represents the number of workers. The only difference is that in standard Neoclassical models $(1-u(t))$ is exogenous and constant and then it does not appear in the analytical resolution.

by a specific equation as in the Keynesian model. In our base model, we introduce a simple macroeconomic investment function as follows:[7]

$$I(t) = \gamma \cdot e^{(n+g_A)t} \tag{4}$$

where γ is positive and may be interpreted as a parameter reflecting animal spirits.

Then, the macroeconomic equilibrium condition becomes:

$$s \cdot \hat{k}(t)^{\alpha} \cdot A(t) \cdot L(t) \cdot (1 - u(t))^{1-\alpha} = \gamma \cdot e^{(n+g_A)t}$$

Then:

$$1 - u(t) = \left(\frac{\gamma}{s}\right)^{\frac{1}{1-\alpha}} \cdot \hat{k}(t)^{-\frac{\alpha}{1-\alpha}} \tag{5}$$

Equation 5 determines the *equilibrium unemployment rate* at each instant t which represents the only value that guarantees the equilibrium between investments and aggregate savings and then the equilibrium between aggregate supply, $Y(t)$, and aggregate demand, $C(t) + I(t)$.

Equation 5 shows that the equilibrium unemployment rate depends on the value of γ. In particular, if $\gamma = s \cdot \hat{k}(t)^{\alpha}$, the unemployment rate is equal to zero. In fact, by considering equation 4 and the definition of $\hat{k}$, $\gamma = s \cdot \hat{k}(t)^{\alpha}$ implies that $I(t) = s \cdot K(t)^{\alpha} \cdot [A(t) \cdot L(t))]^{1-\alpha}$, i.e. investments are equal to private savings in a full-employment economy. In contrast, if $\gamma < s \cdot \hat{k}(t)^{\alpha}$, then the unemployment rate is positive. This means that if γ is lower than the value that allows to achieve full-employment, the economy is in a situation of under-employment equilibrium due to the weakness of the aggregate demand and in particular of investments. It is also important to note that if $\gamma < s \cdot \hat{k}(t)^{\alpha}$ and if the real wage is determined to achieve full-employment (i.e. point A in Figures 1 and 2), then private savings $s \cdot K(t)^{\alpha} \cdot [A(t) \cdot L(t))]^{1-\alpha}$ are greater than investments $\gamma \cdot e^{(n+g_A)t}$, implying that the economy *is not* in equilibrium.

Finally, equation 5 implies that $\frac{\partial u(t)}{\partial \gamma} < 0$, $\frac{\partial u(t)}{\partial s} > 0$, and $\frac{\partial u(t)}{\partial \hat{k}(t)} > 0$. The equilibrium unemployment rate decreases if the parameter that affects investments increases and if the saving rate decreases, since they both induce an increase in the aggregate demand. Moreover, an increase in the capital per unit of effective potential labor provokes an increase in the unemployment rate. Consider the case of an under-capitalized economy in which $\gamma < s \cdot \hat{k}(t)^{\alpha}$, implying that $u(t) > 0$. The fact that the economy is under-capitalized ($\hat{k}(0) < \hat{k}^{*}$) implies that the level of savings per unit of effective potential labor is low and, in order to equalize aggregate savings and investments, the unemployment rate must be sufficiently low. During the transition phase, as analyzed in the following section, the capital per unit of effective potential labor increases implying that the level of savings per unit of effective potential labor increases and, in order to equalize aggregate savings and investments, the unemployment rate also increases.

[7]In the Appendix we present and solve a more general model in which investments also depend (negatively) on the level of the interest rate r. More precisely, we use $I(t) = \gamma \cdot e^{(n+g_A)t} \cdot r(t)^{-\beta}$ with $\beta > 0$. Here we use a simpler expression since in most of the models presented in our paper it is possible to find a closed solution only by fixing $\beta = 0$.

3.2. The steady state and the transition phase towards the long-run equilibrium

The evolution of the capital per unit of effective potential labor is given by:

$$\dot{\hat{k}}(t) = \frac{d\hat{k}(t)}{dt} = \frac{d\left(\frac{K(t)}{A(t)\cdot L(t)}\right)}{dt} = \frac{\dot{K}(t)\cdot A(t)\cdot L(t) - K(t)\cdot(\dot{A}(t)\cdot L(t) + A(t)\cdot \dot{L}(t))}{(A(t)\cdot L(t))^2}$$

The aggregate capital stock evolves according to: $\dot{K}(t) = I(t) - \delta \cdot K(t)$. Then:

$$\dot{\hat{k}}(t) = s\cdot \hat{y}(t) - (n + g_A + \delta)\cdot \hat{k}(t)$$

By considering that $\hat{y}(t) = \hat{k}(t)^{\alpha}\cdot(1-u(t))^{1-\alpha}$ and by taking into account equation 5, we find:

$$\dot{\hat{k}}(t) = s\cdot \hat{k}(t)^{\alpha}\cdot \frac{\gamma}{s}\cdot \hat{k}(t)^{-\alpha} - (n + g_A + \delta)\cdot \hat{k}(t)$$

The dynamics of the capital per unit of effective potential labor is then described by:

$$\dot{\hat{k}}(t) = \gamma - (n + g_A + \delta)\cdot \hat{k}(t) \tag{6}$$

The steady-state condition $\dot{\hat{k}}(t) = 0$ allows us to determine the stationary value of the capital per unit of effective potential labor:

$$\hat{k}^* = \frac{\gamma}{n + g_A + \delta} \tag{7}$$

By considering equation 5 and the stationary value of the capital per unit of effective potential labor, we can determine the stationary value of the unemployment rate u^*:

$$1 - u^* = \frac{\gamma}{s^{\frac{1}{1-\alpha}}}\cdot (n + g_A + \delta)^{\frac{\alpha}{1-\alpha}} \tag{8}$$

Consider an under-developed economy, i.e. an economy in which the initial capital per unit of effective potential labor is lower than its stationary value, i.e. $\hat{k}(0) < \hat{k}^*$. In the transition phase, the capital per unit of effective potential labor increases over time and the unemployment rate increases over time, until the economy reaches its steady state. In particular, the long-run unemployment rate is equal to zero, i.e. the economy converges towards the full-employment long-run equilibrium, only if $\gamma = \frac{s^{\frac{1}{1-\alpha}}}{(n+g_A+\delta)^{\frac{\alpha}{1-\alpha}}}$. In contrast, if γ is lower than this value, then the economy displays unemployment even in the long run.

3.3. The effect of a change in the saving rate

Consider now an increase in the saving rate. If the increase in the saving rate is not accompanied by an increase in the parameter γ, the steady state value of the capital per unit of effective potential labor remains unchanged ($\frac{\partial \hat{k}^*}{\partial s} = 0$), while the unemployment rate increases ($\frac{\partial (1-u^*)}{\partial s} < 0$).

The direct effect of an increase in the saving rate is a reduction in private consumption and, if investments do not increase to compensate the reduction in private consumption, the short run effect is a reduction in the aggregate demand and in real GDP, and then an increase in unemployment. If we consider the dynamics, the increase in the saving rate permits a greater capital accumulation. However, the unemployment rate remains greater than before the shock. By considering that real GDP is given by $Y(t) = \hat{k}(t)^{\alpha} \cdot (1-u(t))^{1-\alpha} \cdot A(t) \cdot L(t)$, the short-run effect is negative since $\hat{k}(t)$ is a predetermined variable and $u(t)$ increases. The long-run effect is also negative since a change in the saving rate has no effect on $\hat{k}^*$ (see equation 7) and increases the unemployment rate (see equation 8). Of course, this result is due to the fact that the specification of the investment function (see equation 4) implies that an increase in the saving rate produces no effect on investments. This assumption is relaxed in the next section.

4. Introduction of a crowding-in/crowding-out effect on investments

In this section, we consider the possibility that a change in private consumption and savings affects investments. We modify the macroeconomic investment function (introduced in equation 4) by adding a term permitting to consider the crowding-in/crowding-out effect on investments:

$$I(t) = \gamma \cdot e^{(n+g_A)t} + \alpha_1 \cdot \Delta S_H(t) \tag{9}$$

where α_1 is a parameter between 0 and 1 that measures the degree of the crowding-in/crowding-out effect on investments. $\Delta S_H(t)$ represents the change in private savings with respect to the situation before a shock.

Consider an increase in the saving rate. In Neoclassical models, the increase in private savings produces a positive effect (and equal in magnitude) on investments. In Keynesian models, in which investments are defined as in equation 4, the same shock produces no effects on investments. The term α_1 allows us to take into account for an infinite number of intermediate situations in which the increase in private savings induces (i) a zero crowding-in effect on investments, as in Keynesian models, implying that $\alpha_1 = 0$, (ii) a complete crowding-in effect on investments, as in Neoclassical models, implying that $\alpha_1 = 1$, and (ii) a partial crowding-in effect on investments which allows to build a model that is between the Keynesian and the Neoclassical model, and implying that $0 < \alpha_1 < 1$. Symmetrically, if the shock is represented by a reduction in the saving rate, the crowding-out effect on investments can be nil, complete, or partial, according to the value of α_1.

In this section we analyze the effects of a change in the saving rate from a value noted s_{old} to a value noted s_{new}. We suppose that, before the shock, the economy is in a situation of steady state. Then, the change in private savings can be computed as follows:

$$\begin{aligned} \Delta S_H(t) &= A(t)\cdot L(t)\cdot\left[s_{new}\cdot\hat{k}(t)^{\alpha}-(n+g_A+\delta)\cdot\hat{k}(t)\right]\cdot(1-u(t))^{1-\alpha} \\ &- A(t)\cdot L(t)\cdot\left[s_{old}\cdot(\hat{k}^*)^{\alpha}-(n+g_A+\delta)\cdot\hat{k}(t)\right]\cdot(1-u^*)^{1-\alpha} \end{aligned}$$

The investment function can be rewritten as:

$$\begin{aligned} I(t) &= \gamma\cdot e^{(n+g_A)t} \\ &+ \alpha_1\cdot A(t)\cdot L(t)\cdot\left[s_{new}\cdot\hat{k}(t)^{\alpha}-(n+g_A+\delta)\cdot\hat{k}(t)\right]\cdot(1-u(t))^{1-\alpha} \\ &- \alpha_1\cdot A(t)\cdot L(t)\cdot\left[s_{old}\cdot(\hat{k}^*)^{\alpha}-(n+g_A+\delta)\cdot\hat{k}(t)\right]\cdot(1-u^*)^{1-\alpha} \end{aligned}$$

4.1. Instantaneous equilibrium

The macroeconomic equilibrium condition, $s_{new}\cdot Y(t)=I(t)$, becomes:

$$\begin{aligned} s_{new}\cdot\hat{k}(t)^{\alpha}\cdot e^{(n+g_A)t}\cdot(1-u(t))^{1-\alpha} &= \gamma\cdot e^{(n+g_A)t} \\ &+ \alpha_1\cdot A(t)\cdot L(t)\cdot\left[s_{new}\cdot\hat{k}(t)^{\alpha}-(n+g_A+\delta)\cdot\hat{k}(t)\right]\cdot(1-u(t))^{1-\alpha} \\ &- \alpha_1\cdot A(t)\cdot L(t)\cdot\left[s_{old}\cdot(\hat{k}^*)^{\alpha}-(n+g_A+\delta)\cdot\hat{k}(t)\right]\cdot(1-u^*)^{1-\alpha} \end{aligned}$$

Then:

$$\begin{aligned} &\left[(1-\alpha_1)\cdot s_{new}\cdot\hat{k}(t)^{\alpha}+\alpha_1\cdot(n+g_A+\delta)\cdot\hat{k}(t)\right]\cdot(1-u(t))^{1-\alpha} \\ =\ &\gamma-\alpha_1\cdot\left[s_{old}\cdot(\hat{k}^*)^{\alpha}\cdot(1-u^*)^{1-\alpha}-(n+g_A+\delta)\cdot\hat{k}(t)\cdot(1-u^*)^{1-\alpha}\right] \end{aligned}$$

By considering that $\hat{k}^*=\frac{\gamma}{n+g_A+\delta}$ (see equation 7) and $1-u^*=\frac{\gamma}{s_{old}^{\frac{1}{1-\alpha}}}\cdot(n+g_A+\delta)^{\frac{\alpha}{1-\alpha}}$ (see equation 8), implying that $(\hat{k}^*)^{\alpha}\cdot(1-u^*)^{1-\alpha}=\left(\frac{\gamma}{n+g_A+\delta}\right)^{\alpha}\cdot\frac{\gamma^{1-\alpha}}{s_{old}}\cdot(n+g_A+\delta)^{\alpha}=\frac{\gamma}{s_{old}}$, we find that the instantaneous equilibrium unemployment rate is given by:

$$1-u(t)=\left[\frac{\gamma-\alpha_1\cdot\left[\gamma-(n+g_A+\delta)\cdot\hat{k}(t)\cdot(1-u^*)^{1-\alpha}\right]}{(1-\alpha_1)\cdot s_{new}\cdot\hat{k}(t)^{\alpha}+\alpha_1\cdot(n+g_A+\delta)\cdot\hat{k}(t)}\right]^{\frac{1}{1-\alpha}}$$

Two extreme cases are interesting: the case $\alpha_1=0$ implying that the crowding-in effect on investments is nil, and the case $\alpha_1=1$ implying that the crowding-in effect on investments is complete:

$$1-u(t)=\begin{cases}\left(\frac{\gamma}{s_{new}}\right)^{\frac{1}{1-\alpha}}\cdot\hat{k}(t)^{-\frac{\alpha}{1-\alpha}} & if\quad \alpha_1=0 \\ 1-u^* & if\quad \alpha_1=1\end{cases} \tag{10}$$

Note that the two polar cases reproduce, respectively, the Keynesian model presented in the previous section and the Neoclassical model in which the unemployment rate is exogenous and is not affected by any shock. Then, the previous expression implies that an increase in the saving rate increases the unemployment rate, excepted the case in which the crowding-in effect is complete ($\alpha_1 = 1$). In addition, the magnitude of the (negative) effect of such a shock is a decreasing function of α_1.

4.2. The steady state

As in the base model, the evolution of the capital per unit of effective potential labor is given by $\dot{\hat{k}}(t) = s_{new} \cdot \hat{y}(t) - (n + g_A + \delta) \cdot \hat{k}(t)$. By considering that $\hat{y}(t) = \hat{k}(t)^{\alpha} \cdot (1 - u(t))^{1-\alpha}$ and by considering Equation 10, we find:

$$\dot{\hat{k}}(t) = s_{new} \cdot \hat{k}(t)^{\alpha} \cdot \left[\frac{\gamma - \alpha_1 \cdot \left[\gamma - (n+g_A+\delta) \cdot \hat{k}(t) \cdot (1-u^*)^{1-\alpha}\right]}{(1-\alpha_1) \cdot s_{new} \cdot \hat{k}(t)^{\alpha} + \alpha_1 \cdot (n+g_A+\delta) \cdot \hat{k}(t)} \right] - (n+g_A+\delta) \cdot \hat{k}(t)$$

In particular, in the two polar cases, the evolution of the capital per unit of effective potential labor is given by:

$$\dot{\hat{k}}(t) = \begin{cases} \gamma - (n+g_A+\delta) \cdot \hat{k}(t) & if \quad \alpha_1 = 0 \\ s_{new} \cdot \hat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha} - (n+g_A+\delta) \cdot \hat{k}(t) & if \quad \alpha_1 = 1 \end{cases} \tag{11}$$

The steady-state condition $\dot{\hat{k}}(t) = 0$ allows us to determine the new stationary value of the capital per unit of effective potential labor. In particular, the long-term value of the capital per unit of effective potential labor in the two polar cases is:

$$\hat{k}^{**} = \begin{cases} \frac{\gamma}{n+g_A+\delta} & if \quad \alpha_1 = 0 \\ \left(\frac{s_{new}}{n+g_A+\delta}\right)^{\frac{1}{1-\alpha}} \cdot (1-u^*) & if \quad \alpha_1 = 1 \end{cases} \tag{12}$$

This implies that the capital per unit of effective potential labor is not affected by the increase in the saving rate in the case in which the crowding-in effect is nil, as we have already found in the previous section. However, the effect is positive when the crowding-in effect is complete, as in Neoclassical models, but also in the case in which the crowding-in effect is partial, i.e. $0 < \alpha_1 < 1$.

By considering again equation 10 and the stationary value of the capital per unit of effective potential labor (equation 12), we can determine the new stationary value of the unemployment rate for the two polar cases:

$$1 - u^{**} = \begin{cases} \frac{\gamma}{s_{new}^{\frac{1}{1-\alpha}}} \cdot (n+g_A+\delta)^{\frac{\alpha}{1-\alpha}} & if \quad \alpha_1 = 0 \\ 1 - u^* & if \quad \alpha_1 = 1 \end{cases} \tag{13}$$

An increase in the saving rate increases the steady state unemployment rate, $u^{**} > u^*$, excepted the case in which the crowding-in effect is complete, i.e. $\alpha_1 = 1$.

Consider now the effect on the real GDP. By considering that real GDP is given by $Y(t) = \hat{k}(t)^\alpha \cdot (1-u(t))^{1-\alpha} \cdot A(t) \cdot L(t)$, the GDP level at the moment in which the saving rate increases ($t = 0$), for the two polar cases, is given by:

$$Y(0) = \begin{cases} \frac{\gamma}{s_{new}} \cdot A(0) \cdot L(0) & if \quad \alpha_1 = 0 \\ \hat{k}(0)^\alpha \cdot (1-u^*)^{1-\alpha} \cdot A(0) \cdot L(0) & if \quad \alpha_1 = 1 \end{cases} \tag{14}$$

This means that, with the exception of the case $\alpha_1 = 1$, i.e. the case in which the crowding-in effect on investments is complete, the short-run effect is negative since $\hat{k}(t)$ is a predetermined variable and unemployment increases. In contrast, if $\alpha_1 = 1$, there is no effect on real GDP in the short-run, since the unemployment rate is not affected.

In the long-run, the GDP level is given by:

$$Y(t) = \begin{cases} \frac{\gamma}{s_{new}} \cdot A(t) \cdot L(t) & if \quad \alpha_1 = 0 \\ \left(\frac{s_{new}}{n+g_A+\delta}\right)^{\frac{\alpha}{1-\alpha}} \cdot (1-u^*) \cdot A(t) \cdot L(t) & if \quad \alpha_1 = 1 \end{cases} \tag{15}$$

The effect on the long-run value of GDP of an increase in the saving rate is negative if $\alpha_1 = 0$ and positive if $\alpha_1 = 1$. This also means that there exists a threshold value of α_1 such that if the parameter measuring the crowding-in effect is greater than the threshold value, the effect on the long-run GDP is positive, while, if the parameter measuring the crowding-in effect is lower than the threshold value, the effect on the long-run GDP is negative. Note also that if $\alpha_1 = 0$, an increase in the saving rate produces the same negative effect on GDP in the short and in the long run.

5. Introduction of public expenditures

Now we assume that, starting from a situation of steady state, the government introduces expenditures G. Public expenditures are assumed to increase over time at the rate $n+g$, implying that the public expenditure per unit of effective potential labor $\hat{g}$ is constant.[8] This shock is assumed to be permanent and unanticipated. Of course, the government has to introduce taxes such that the present value of all the taxes equals the present value of public expenditures. The easiest way to introduce in our model the taxes in order to respect the intertemporal budget constraint of the government is to assume that the government introduces a lump-sum tax such that, at each instant t, $T(t) = G(t)$, implying that public savings are always equal to zero.

[8] In fact, $G(t) = G(0) \cdot e^{(n+g_A)t}$ implies that $\frac{G(t)}{A(t) \cdot L(t)} \equiv \hat{g} = G(0)$.

5.1. Instantaneous equilibrium

The macroeconomic equilibrium condition states that investments must be equal to aggregate savings:

$$I(t) = s \cdot [Y(t) - G(t)]$$

By using the investment function defined in equation 4, we find:

$$s \cdot \left[\hat{k}(t)^{\alpha} \cdot (1-u(t))^{1-\alpha} - \hat{g}\right] \cdot A(t) \cdot L(t) = \gamma \cdot e^{(n+g_A)t}$$

Then, the *equilibrium unemployment rate* at each instant t is given by:

$$1 - u(t) = \left(\frac{\gamma}{s} + \hat{g}\right)^{\frac{1}{1-\alpha}} \cdot \hat{k}(t)^{-\frac{\alpha}{1-\alpha}} \tag{16}$$

The previous expression implies that the *equilibrium unemployment rate* depends negatively on the value of $\hat{g}$. This implies that at the moment in which the government introduces (or, more generally, increases) public expenditures and lump-sum taxes, and given that the capital per unit of effective potential labor is predetermined, then the unemployment rate decreases thanks to the implementation of such an expansionary fiscal policy.

5.2. The steady state

In the presence of public expenditures and lump-sum taxes as previously described, the evolution of the capital per unit of effective potential labor is given by $\dot{\hat{k}}(t) = s \cdot [\hat{y}(t) - \hat{g}] - (n + g_A + \delta) \cdot \hat{k}(t)$. By considering that $\hat{y}(t) = \hat{k}(t)^{\alpha} \cdot (1-u(t))^{1-\alpha}$ and by considering equation 16, we find:

$$\dot{\hat{k}}(t) = s \cdot \left[\hat{k}(t)^{\alpha} \cdot \left(\frac{\gamma}{s} + \hat{g}\right) \cdot \hat{k}(t)^{-\alpha} - \hat{g}\right] - (n + g_A + \delta) \cdot \hat{k}(t)$$

The dynamics of the capital per unit of effective potential labor is then described by:

$$\dot{\hat{k}}(t) = [\gamma - (1-s) \cdot \hat{g}] - (n + g_A + \delta) \cdot \hat{k}(t) \tag{17}$$

The steady-state condition $\dot{\hat{k}}(t) = 0$ allows us to determine the stationary value of the capital per unit of effective potential labor:

$$\hat{k}^{*} = \frac{\gamma - (1-s) \cdot \hat{g}}{n + g_A + \delta} \tag{18}$$

This implies that an increase in public expenditures induces a reduction in the steady state value of the capital per unit of effective potential labor.

By considering again equation 16 and the stationary value of the capital per unit of effective potential labor (equation 18), we can determine the stationary value of the unemployment rate u^*:

$$1 - u^* = \left(\frac{\gamma}{s} + \hat{g}\right)^{\frac{1}{1-\alpha}} \cdot \left(\frac{\gamma - (1-s)\cdot\hat{g}}{n + g_A + \delta}\right)^{-\frac{\alpha}{1-\alpha}} \tag{19}$$

This implies that an increase in public expenditures allows to reduce the long-term level of the unemployment rate: $\frac{\partial(1-u^*)}{\partial \hat{g}} > 0$.

By assuming that investments are not affected by the introduction (or, more generally, the increase in the level) of public expenditures, in the short run, the aggregate demand increases, real GDP increases and unemployment decreases. If we consider the dynamics, this expansionary fiscal policy reduces capital accumulation. However, the unemployment rate remains lower than before the shock. By considering that GDP is given by $Y(t) = \hat{k}(t)^{\alpha} \cdot (1-u(t))^{1-\alpha} \cdot A(t) \cdot L(t)$, the short-run effect is positive since $\hat{k}(t)$ is a predetermined variable and $u(t)$ decreases. The long-run effect of the introduction of an expansionary fiscal policy is also positive. In fact:

$$\begin{aligned} Y(t) &= (\hat{k}^*)^{\alpha} \cdot (1-u^*)^{1-\alpha} \cdot A(t) \cdot L(t) \\ &= \left(\frac{\gamma - (1-s)\cdot\hat{g}}{n + g_A + \delta}\right)^{\alpha} \cdot \left[\left(\frac{\gamma}{s} + \hat{g}\right)^{\frac{1}{1-\alpha}} \cdot \left(\frac{\gamma - (1-s)\cdot\hat{g}}{n + g_A + \delta}\right)^{-\frac{\alpha}{1-\alpha}}\right]^{1-\alpha} \cdot A(t) \cdot L(t) \\ &= \left(\frac{\gamma}{s} + \hat{g}\right) \cdot A(t) \cdot L(t) \end{aligned}$$

This implies that the long-run value of GDP is stimulated when an expansionary fiscal policy is introduced. This result is due to the fact that the model is demand-driven and the specification of the investment function (equation 4) implies that an increase in public expenditures produces no crowding-out effect on investments. Of course, in Neoclassical models, the effect of the same fiscal policy is completely different: an expansionary fiscal policy reduces aggregate savings and investments; this produces a negative effect on capital accumulation and then on GDP. Our hypothesis that an increase in public expenditures produces no crowding-out effect on investments is relaxed in the next section.

6. Introduction of public expenditures with (partial) crowding-out effect on investments

As in the previous section, we assume that the government introduces expenditures and a lump-sum tax such that $T(t) = G(t)$.

Now, we write the investment function as follows:

$$I(t) = \gamma \cdot e^{(n+g_A)t} + \alpha_1 \cdot [\Delta S_H(t) + \Delta S_G(t)] \tag{20}$$

where α_1 is again a parameter between 0 and 1 that measures the degree of the crowding-in/crowding-out effect on investments, $\Delta S_H(t)$ represents the change in private savings with respect to the situation before a shock, and $\Delta S_G(t)$ represents the change in public savings with respect to the situation before a shock. The investment function defined in equation 20 allows to take into account for the crowding-out effect provoked by an increase in public expenditures.

Starting from a situation of steady state, the introduction of public expenditures, accompanied by the introduction of a lump-sum tax such that $T(t) = G(t)$, has no effect on public savings ($\Delta S_G(t) = 0$) and produces the following change in private savings:

$$\begin{aligned}\Delta S_H(t) &= \left[s\cdot\left(\hat{k}(t)^{\alpha}\cdot(1-u(t))^{1-\alpha}-\hat{g}\right)-(n+g_A+\delta)\cdot\hat{k}(t)\right]\cdot A(t)\cdot L(t)\\ &- \left[s\cdot(\hat{k}^{*})^{\alpha}-(n+g_A+\delta)\cdot\hat{k}(t)\right]\cdot(1-u^{*})^{1-\alpha}\cdot A(t)\cdot L(t)\end{aligned}$$

6.1. Instantaneous equilibrium

The macroeconomic equilibrium condition can be written as follows:

$$\begin{aligned}s\cdot\left[\hat{k}(t)^{\alpha}\cdot(1-u(t))^{1-\alpha}-\hat{g}\right] &= \gamma\\ &+ \alpha_1\cdot\left[s\cdot\left(\hat{k}(t)^{\alpha}\cdot(1-u(t))^{1-\alpha}-\hat{g}\right)-(n+g_A+\delta)\cdot\hat{k}(t)\right]\\ &- \alpha_1\cdot\left[s\cdot(\hat{k}^{*})^{\alpha}-(n+g_A+\delta)\cdot\hat{k}(t)\right]\cdot(1-u^{*})^{1-\alpha}\end{aligned}$$

Then:

$$\begin{aligned}&\left[(1-\alpha_1)\cdot s\cdot\hat{k}(t)^{\alpha}+\alpha_1\cdot(n+g_A+\delta)\cdot\hat{k}(t)\right]\cdot(1-u(t))^{1-\alpha}\\ =\ &\gamma+s\cdot(1-\alpha_1)\cdot\hat{g}-\alpha_1\cdot\left[s\cdot(\hat{k}^{*})^{\alpha}\cdot(1-u^{*})^{1-\alpha}-(n+g_A+\delta)\cdot\hat{k}(t)\cdot(1-u^{*})^{1-\alpha}\right]\end{aligned}$$

By considering that, without public expenditures, $\hat{k}^{*} = \frac{\gamma}{n+g_A+\delta}$ (see equation 7) and $1-u^{*} = \frac{\gamma}{s^{\frac{1}{1-\alpha}}}\cdot(n+g_A+\delta)^{\frac{\alpha}{1-\alpha}}$ (see equation 8), implying that $(\hat{k}^{*})^{\alpha}\cdot(1-u^{*})^{1-\alpha} = \frac{\gamma}{s}$, we find that the instantaneous equilibrium unemployment rate is given by:

$$1-u(t) = \left[\frac{\gamma+s\cdot(1-\alpha_1)\cdot\hat{g}-\alpha_1\cdot\left[\gamma-(n+g_A+\delta)\cdot\hat{k}(t)\cdot(1-u^{*})^{1-\alpha}\right]}{(1-\alpha_1)\cdot s\cdot\hat{k}(t)^{\alpha}+\alpha_1\cdot(n+g_A+\delta)\cdot\hat{k}(t)}\right]^{\frac{1}{1-\alpha}} \tag{21}$$

In particular, the unemployment rate in the two polar cases is:

$$1-u(t) = \begin{cases}\left(\frac{\gamma}{s}+\hat{g}\right)^{\frac{1}{1-\alpha}}\cdot\hat{k}(t)^{\frac{\alpha}{1-\alpha}} & if\quad \alpha_1 = 0\\ 1-u^{*} & if\quad \alpha_1 = 1\end{cases} \tag{22}$$

The previous expression implies that the introduction of public expenditures, accompanied by a simultaneous introduction of a lump-sum tax, permits to reduce the level of unemployment, excepted in the case $\alpha_1 = 1$.

6.2. The steady state

The introduction of public expenditures and lump-sum taxes as previously described, implies that the evolution of the capital per unit of effective potential labor is given by $\dot{\hat{k}}(t) = s \cdot [\hat{y}(t) - \hat{g}] - (n + g_A + \delta) \cdot \hat{k}(t)$. By considering that $\hat{y}(t) = \hat{k}(t)^{\alpha} \cdot (1 - u(t))^{1-\alpha}$ and by considering equation 21, the dynamics of the capital per unit of effective potential labor is described by:

$$\begin{aligned} \dot{\hat{k}}(t) &= s \cdot \left[\hat{k}(t)^{\alpha} \cdot \frac{\gamma + s \cdot (1-\alpha_1) \cdot \hat{g} - \alpha_1 \cdot \left[\gamma - (n + g_A + \delta) \cdot \hat{k}(t) \cdot (1-u^*)^{1-\alpha} \right]}{(1-\alpha_1) \cdot s \cdot \hat{k}(t)^{\alpha} + \alpha_1 \cdot (n + g_A + \delta) \cdot \hat{k}(t)} - \hat{g} \right] \\ &- (n + g_A + \delta) \cdot \hat{k}(t) \end{aligned}$$

In particular, in the two polar cases, the evolution of the capital per unit of effective potential labor is given by:

$$\dot{\hat{k}}(t) = \begin{cases} (\gamma + (1-s) \cdot \hat{g}) - (n + g_A + \delta) \cdot \hat{k}(t) & if \quad \alpha_1 = 0 \\ s \cdot \left[\hat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha} - \hat{g} \right] - (n + g_A + \delta) \cdot \hat{k}(t) & if \quad \alpha_1 = 1 \end{cases} \tag{23}$$

The steady-state condition, $\dot{\hat{k}}(t) = 0$, allows us to determine the new stationary value of the capital per unit of effective potential labor. It is possible to determine a closed solution for the long-term value of the capital per unit of effective potential labor only if $\alpha_1 = 0$. In any cases, the long-term value of the capital per unit of effective potential labor is negatively affected by the introduction of public expenditures. In the case of $\alpha_1 = 0$, the new stationary value of the capital per unit of effective potential labor is:

$$\hat{k}^{**} = \left\{ \frac{\gamma - (1-s) \cdot \hat{g}}{n + g_A + \delta} \quad if \quad \alpha_1 = 0 \right. \tag{24}$$

By considering again equation 21 and the stationary value of the capital per unit of effective potential labor (equation 24), we can determine the new stationary value of the unemployment rate for the two polar cases:

$$1 - u^{**} = \begin{cases} \frac{\gamma - \cdot \hat{g}}{s^{\frac{1}{1-\alpha}}} \cdot (n + g_A + \delta)^{\frac{\alpha}{1-\alpha}} & if \quad \alpha_1 = 0 \\ 1 - u^* & if \quad \alpha_1 = 1 \end{cases} \tag{25}$$

This implies that the expansionary fiscal policy permits a reduction of the long-term unemployment rate, excepted in the case $\alpha_1 = 1$.

7. Numerical simulations

In this section we present numerical simulations in order to analyze the evolution of (i) an under-developed economy, (ii) of an economy in which the saving rate increases, and (iii) an economy in which public expenditures are introduced.

We first calibrate our model at the steady state without public expenditures and taxes. Our economy is characterized by a population growth rate (n) of 0.5%, a productivity growth rate (g) of 1.5%, and a depreciation rate (δ) of 4%. Moreover, α in the Cobb-Douglas production function is fixed at $1/3$, the saving rate (s) is equal to 20% and the parameter γ in the investment equation has been calibrated in order to obtain a stationary unemployment rate equal to 10%.

7.1. Transition of an under-capitalized economy

In the first simulation, we assume that the initial capital per unit of effective labor is equal to 60% of the long-run value, implying that the economy is under-capitalized. Figure 5 shows the economic transition towards the steady state. In particular, the evolution of the capital per unit of effective labor and per unit of effective potential labor is presented in Figure 5a, the rate of growth of real GDP in Figure 5b, and the unemployment rate in Figure 5c. The simulation results indicate that the under-developed economy converges towards the steady state. In particular, the unemployment rate increases over time from an initial value of 5.8% to the long-run value of 10%. One interesting aspect of this simulation is the relation between the growth rate of real wages and the unemployment rate. Figure 5d shows the negative relation between these two variables that is coherent with the Phillips curve.[9]

7.2. Increase in the saving rate

In the second simulation we assume that the economy is already at the steady state and that the private saving rate increases from 20% to 21%.

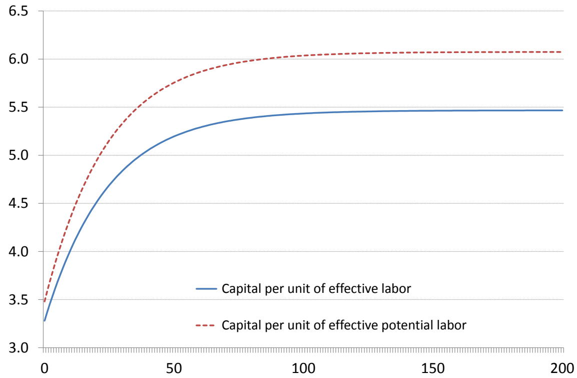
We first solve the model using the Solow model, i.e. by assuming that investments are determined by aggregate savings instead of by the investment function defined in equation 4 and by fixing the unemployment rate at 10% or, equivalently, by assuming that the number of workers is equal, at each period, to 90% of the active population. Then, we solve the model by introducing equation 4 and by endogenizing the unemployment rate. Finally, we solve the model by considering different values of α_1, i.e. different degrees of the crowding-in/crowding-out effect on investments.

The results are reported in Figure 6. In particular, Figure 6a shows the effect on the unemployment rate. In the Keynesian model, i.e. with $\alpha_1 = 0$, the increase in the saving rate produces a

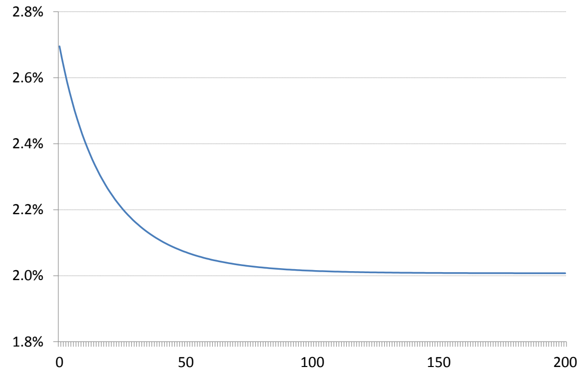
[9] The curve presented in Figure 5d coincides with the traditional Phillips curve only if the inflation rate is equal to zero. However, if the inflation rate is constant or sufficiently stable over time, the curve representing the relation between the growth rate of nominal wages and the unemployment rate is the same as the one depicted in Figure 5d, with the only difference that it is shifted upward.

Figure 5 – Economic transition towards the steady state

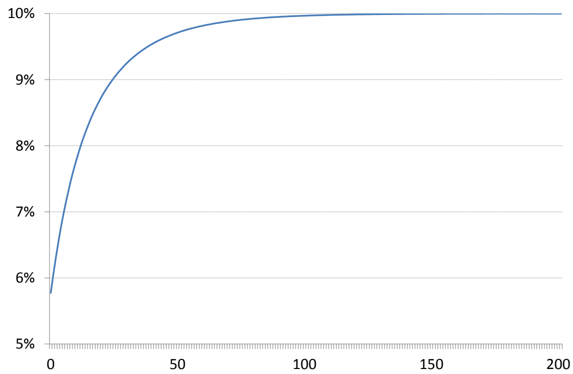
(a) Evolution of $\hat{k}(t)$ and $\tilde{k}(t)$



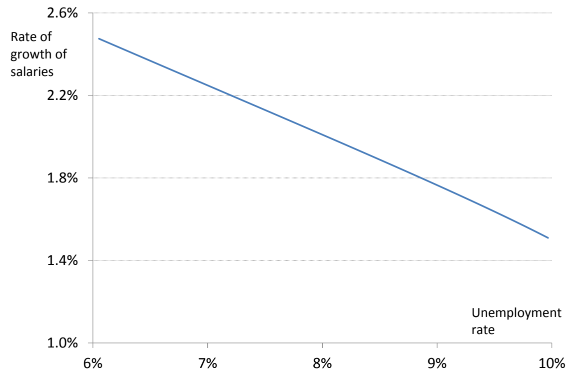
(b) Evolution of the GDP growth rate



(c) Evolution of the unemployment rate



(d) Relation between the wage growth rate and the unemployment rate

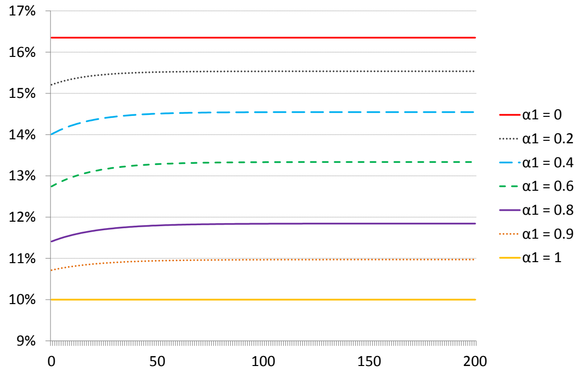


strong increase in the unemployment rate since this shock induces a reduction in private consumption and then in aggregate demand. In particular, the unemployment rate increases to 16.4%. The negative effect on unemployment is less important if we take into account for the crowding-in effect on investments. For example, in the case in which α_1 is equal to 0.2, the unemployment rate becomes equal to 15.2% in the short run and equal to 15.5% in the long run. In addition, a more important value of α_1 implies a lower negative impact on the unemployment rate. In particular, in the case in which $\alpha_1 = 1$, the reduction in private consumption is perfectly compensated by the increase in investments. This implies that the aggregate demand remains unchanged and the unemployment rate remains equal to 10% as before the shock, as in the Solow model.

Figure 6b shows the effect on GDP, measured as the percentage deviations with respect to the situation before the shock. The most negative effect is produced in the Keynesian model, i.e. with $\alpha_1 = 0$, where the value of GDP is 4.8% lower than before the shock. The negative shock is reduced if α_1 is positive. Interestingly, if α_1 is equal to 0.8 and to 0.9, the short run effect

Figure 6 – Economic impacts of an increase in the saving rate

(a) Evolution of the unemployment rate



(b) Evolution of GDP (% deviations to the baseline)

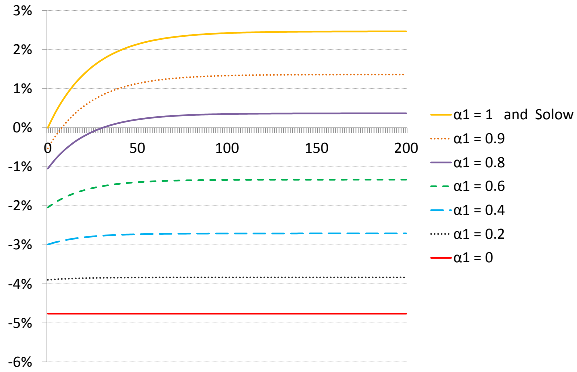
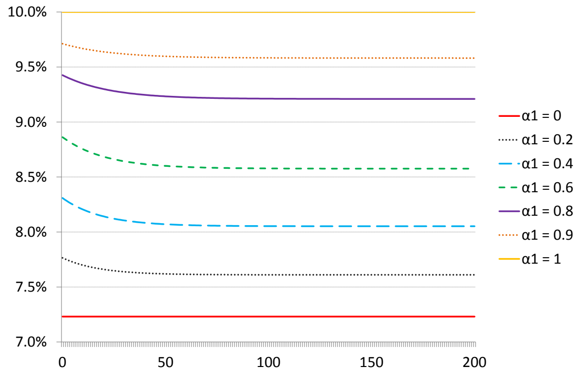
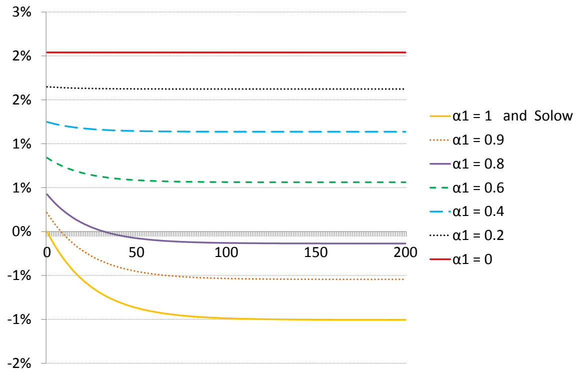


Figure 7 – Economic impacts of the introduction of public expenditures

(a) Evolution of the unemployment rate



(b) Evolution of GDP (% deviations to the baseline)



on GDP is negative in the short run (-1%) and becomes positive after some periods. In the case in which α_1 is equal to 1, there is no effect on GDP in the short run since unemployment remains unchanged, while the long-run effect is positive (+2.5%). The evolution of the economy obtained in our model with $\alpha_1 = 1$ is exactly the same obtained in the Solow model.

7.3. Introduction of public expenditures

In the last simulation, we assume that the economy is already at the steady state and the government introduces, permanently, public expenditures and lump-sum taxes which both represent 2% of GDP in each period.

Figures 7a and 7b show respectively the effect on the unemployment rate and on GDP. In the Keynesian model, i.e. with $\alpha_1 = 0$, the introduction of the fiscal policy reduces the unemployment rate from 10% to 7.2% and stimulates GDP (+2%), with respect to the situation before the shock. In the Solow model and in our model with $\alpha_1 = 1$, the unemployment remains unchanged, while the GDP is negatively affected by the reform (-1% in the long run with respect to the situation before the shock). As in the simulation of a change in the saving rate, if α_1 is equal to 0.8 and to 0.9, the short run effect on GDP is positive in the short run thanks to the reduction in unemployment, but becomes negative after some periods due to the unfavorable evolution of capital.

8. CONCLUSIONS

The aim of this paper is to extend the Solow model in a way that permits to endogenize unemployment that is caused by the weakness of the aggregate demand. The introduction of Keynesian unemployment in the Solow model is made possible by relaxing the hypothesis, used in the Classical and Neoclassical theory, of full-utilization of the production factors.

Our base model, that with respect the Solow model presents one supplementary equation (the investment function) and one supplementary variable (the unemployment rate), is characterized by the fact that the instantaneous equilibrium may be an *under-employment equilibrium*, implying that involuntary unemployment appears in the case of weakness of the aggregate demand. Using our base model, we show that an under-developed economy converges towards its steady-state equilibrium that may be characterized by a positive value of the unemployment rate and, during the transition phase, the capital per unit of effective labor and the unemployment rate increase over time, until the economy reaches its steady state. Then, using our base model, that works as a Keynesian demand-driven model, we find that an increase in the aggregate demand (due to a reduction in the saving rate or to an increase in public expenditures), reduces unemployment and stimulates the GDP.

Our base model is then extended by modifying the investment function in a way which allows us to take into account for the crowding-in/crowding-out effect on investments. In particular we use a parameter that measures the degree of the crowding-in/crowding-out effect. We show that if this parameter is equal to zero, the model coincides with our base model, i.e. the Keynesian demand-driven model; if the parameter is equal to one, the model coincides with the Solow model and the unemployment rate remains unchanged; if the parameter is between zero and one, the model is an intermediate model which allows to take into account that a shock or a policy, that increases the aggregate demand, stimulates GDP and reduces unemployment (while, in Neoclassical models, the real effect is nil), but also produces a (partial) crowding-out effect on investments (that is not taken into account in Keynesian models). Simulation results show that the effect of a policy or a shock on real GDP may be positive or negative according to the value of α_1 which indicates how much a change in private and public savings affects investments. Therefore, the estimation of α_1 using time-series data represents the essential element of our model.

References

Backhouse, Roger E., (1981), Keynesian Unemployment and the One-Sector Neoclassical Growth Model, *Economic Journal*, 91, issue 361, p. 174-87.

Barro, Robert J. and Grossman, Herschel., (1971), A General Disequilibrium Model of Income and Employment, *American Economic Review*, 61, issue 1, p. 82-93.

Benassy, Jean-Pascal, (1975), Neo-Keynesian Disequilibrium Theory in a Monetary Economy, *Review of Economic Studies*, 42, issue 4, p. 503-23.

Chatterjee, Santanu, (2005), Capital Utilization, Economic Growth and Convergence, *Journal of Economic Dynamics and Control*, 29, issue 12, p. 2093-2124.

Clower, Robert, The Keynesian Counter-Revolution: a Theoretical Appraisal, in Hahn and Brechling, *The Theory of Interest Rates*, Macmillan, London, 1965, p. 103-125.

Davidson, Paul, (1967), A Keynesian View of Patinkin's Theory of Employment, *The Economic Journal*, 77, issue 307, p. 559-578.

Davidson, Paul, (1983), The Marginal Product Curve Is Not the Demand Curve for Labor and Lucas's Labor Supply Function Is Not the Supply Curve for Labor in the Real World, *Journal of Post Keynesian Economics*, 6, Issue 1, p. 105-117.

Keynes, John Maynard, (1936), *The General Theory of Employment*, Interest and Money, New York.

Leijonhufvud, Axel. (1968), *On Keynesian Economics and the Economics of Keynes: A Study in Monetary Theory*, New York: Oxford University Press.

Malinvaud, Edmond (1977), *The Theory of Unemployment Reconsidered*, Oxford, Basil Blackwell.

Mankiw, N. Gregory, (1989), Real Business Cycles: A New Keynesian Perspective, *Journal of Economic Perspectives*, 3, issue 3, p. 79-90.

Patinkin, Don (1965), *Money, Interest and Prices*, 2nd edition. New York.

Solow, Robert, (1956), A Contribution to the Theory of Economic Growth, *The Quarterly Journal of Economics*, 70, issue 1, p. 65-94.

APPENDIX

Here we analyze our base model, i.e. without government and without a crowding-in/crowding-out effect, by considering a more general investment function. Investments are defined by the following expression:

$$I(t) = \gamma \cdot e^{(n+g_A)t} \cdot (r(t)+\delta)^{-\beta} \tag{26}$$

where γ and β are positive parameters, implying that investments negatively depend on the gross rate of remuneration of capital.

By considering that $r(t)+\delta = \alpha \cdot (1-u(t))^{1-\alpha} \cdot \hat{k}(t)^{\alpha-1}$, in an economy without government, the macroeconomic equilibrium between investments and aggregate savings is given by:

$$s \cdot \hat{k}(t)^{\alpha} \cdot A(t) \cdot L(t) \cdot (1-u(t))^{1-\alpha} = \gamma \cdot e^{(n+g_A)t} \cdot \left[\alpha \cdot (1-u(t))^{1-\alpha} \cdot \hat{k}(t)^{\alpha-1}\right]^{-\beta}$$

Then, the *equilibrium unemployment rate* at each instant t is given by:

$$1-u(t) = \left(\frac{\gamma}{s \cdot \alpha^{\beta}}\right)^{\frac{1}{(1+\beta)(1-\alpha)}} \cdot \hat{k}(t)^{\frac{\beta(1-\alpha)-\alpha}{(1+\beta)(1-\alpha)}} \tag{27}$$

Equation 27 implies that $\frac{\partial u(t)}{\partial \gamma} < 0$, $\frac{\partial u(t)}{\partial s} > 0$, and $\frac{\partial u(t)}{\partial \hat{k}(t)} > 0$ with $\beta < \frac{\alpha}{1-\alpha}$.

The evolution of the capital per unit of effective potential labor is given by:

$$\begin{aligned}\dot{\hat{k}}(t) &= \frac{d\hat{k}(t)}{dt} = \frac{d\left(\frac{K(t)}{A(t)\cdot L(t)}\right)}{dt} \\ &= \frac{\dot{K}(t)\cdot A(t)\cdot L(t) - K(t)\cdot(\dot{A}(t)\cdot L(t) + A(t)\cdot \dot{L}(t))}{(A(t)\cdot L(t))^2}\end{aligned}$$

Given that $\dot{\hat{k}}(t) = s \cdot \hat{y}(t) - (n+g_A+\delta)\cdot \hat{k}(t)$, $\hat{y}(t) = \frac{Y(t)}{A(t)\cdot L(t)} = \hat{k}(t)^{\alpha}\cdot(1-u(t))^{1-\alpha}$ and by considering Equation 27, we find:

$$\dot{\hat{k}}(t) = s \cdot \hat{k}(t)^{\alpha} \cdot \left(\frac{\gamma}{s\cdot\alpha^{\beta}}\right)^{\frac{1}{1+\beta}} \cdot \hat{k}(t)^{\frac{\beta(1-\alpha)-\alpha}{1+\beta}} - (n+g_A+\delta)\cdot\hat{k}(t)$$

The dynamics of the capital per unit of effective potential labor is described by:

$$\dot{\hat{k}}(t) = \frac{s^{\frac{\beta}{1+\beta}} \cdot \gamma^{\frac{1}{1+\beta}}}{\alpha^{\frac{\beta}{1+\beta}}} \cdot \hat{k}(t)^{\frac{\beta}{1+\beta}} - (n+g_A+\delta)\cdot\hat{k}(t)$$

The steady-state condition $\dot{\hat{k}}(t) = 0$ allows us to determine the stationary value of the capital per unit of effective potential labor:

$$\hat{k}^* = \frac{s^\beta \cdot \gamma}{\alpha^\beta \cdot (n + g_A + \delta)^{1+\beta}} \tag{28}$$

By considering again equation 27 and the stationary value of the capital per unit of effective potential labor, we can determine the stationary value of the unemployment rate u^*:[10]

$$1 - u^* = \frac{\gamma}{s \cdot \alpha^\beta} \cdot \left(\frac{s}{n + g_A + \delta} \right)^{\beta - \frac{\alpha}{1-\alpha}} \tag{29}$$

The equilibrium exists, is unique,[11] and is stable if $\beta < 1$.

In the transition phase, an under-developed economy with $\hat{k}(0) < \hat{k}^*$, the capital par unit of effective potential labor increase over time and, with β sufficiently low, the unemployment rate increases over time, until the economy reaches its steady state.

[10] See the Appendix for the detailed computation.
[11] Without taking into account for the case $\hat{k}^* = 0$.

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